MAT 150A HW03

[ADD YOUR NAME HERE]

Due Tuesday, 10/24/23 at 11:59 pm on Gradescope

Reminder. *Your homework submission* ***must be typed*** *(TeX'ed) up in full sentences, with proper mathematical formatting. The following resources may be useful as you learn to use TeX and Overleaf:*

- *Overleaf's introduction to LaTeX:*
  https://www.overleaf.com/learn/latex/Learn_LaTeX_in_30_minutes
- *Detexify:*
  https://detexify.kirelabs.org/classify.html

Covered in this HW Parts of Chp. 2, esp. §2.5–2.12. Homomorphisms, isomorphisms, cosets, index of subgroups, the Correspondence Theorem, product groups, quotient groups.

Grading Since Exam 1 is on Wednesday, October 25, all parts-of-problems in this homework will be graded out of 2 points. The goal is to practice and understand as many problems as you can. I advise against spending a disproportionate amount of time on any one problem; ask for help before the exam!

Exercise 1

Let $\varphi : G \to H$ be an *isomorphism*. Prove that for all $g \in G$, the order of g is the same as the order of $\varphi(g)$: $\operatorname{ord}(g) = \operatorname{ord}(\varphi(g))$.

SOLUTION.

Exercise 2

Let $(A, \star)$ and $(B, \diamond)$ be groups, and let $A \times B$ be their direct product. Recall that multiplication (i.e. the group operation, law of composition) is defined by

$$(a_1, b_1)(a_2, b_2) = (a_1 \star a_2, b_1 \diamond b_2)$$

for $a_i \in A$, $b_i \in B$, $i = 1, 2$. In this exercise, you will verify all the group axioms for $A \times B$.

(a) Prove that multiplication is associative.

(b) What's the identity element $A \times B$? (Prove it.)

(c) What's the inverse of $(a, b) \in A \times B$? (Prove it.)

SOLUTION.

Exercise 3

(a) Let p be a prime number. How many automorphisms does the cyclic group C_p have?

(b) How many automorphisms does C_{24} have?

SOLUTION.

Exercise 4

Let K and H be subgroups of a group G.

(a) Prove that the intersection $K \cap H$ is a subgroup of G.

(b) Prove that if $K \trianglelefteq G$, then $K \cap H \trianglelefteq H$.

SOLUTION.

Exercise 5

Prove that in a group, the products ab and ba are conjugate elements.

SOLUTION.

Exercise 6

Prove that every subgroup of index 2 is a normal subgroup.

SOLUTION.