MAT 150A HW05

[add your name here]

Due Tuesday, 11/7/23 at 11:59 pm on Gradescope

Reminder. *Your homework submission* ***must be typed*** *(TeX'ed) up in full sentences, with proper mathematical formatting. The following resources may be useful as you learn to use TeX and Overleaf:*

- *Overleaf's introduction to LaTeX:*
  https://www.overleaf.com/learn/latex/Learn_LaTeX_in_30_minutes
- *Detexify:*
  https://detexify.kirelabs.org/classify.html

Covered in this HW §2.10, 2.12; 3.2; parts of 7.9, 7.10. Correspondence theorem, quotient groups, first isomorphism theorem, generators and relations.

Grading Some of the (parts of) problems will be graded in detail out of several points, and necessary feedback will be given. The rest will be graded out of 2 points.

Remark. Mathematicians sometimes use $*$ to indicate "some value". For example, below, the matrix on the left represents the entire set of 2×2 upper triangular matrices over a field $\mathbb{F}$, on the right:

$$\begin{bmatrix} * & * \\ 0 & * \end{bmatrix} = \left\{ \begin{bmatrix} a & b \\ 0 & c \end{bmatrix} \mid a, b, c \in \mathbb{F} \right\}.$$

Exercise 1

Let G be a group, let $N \trianglelefteq G$, and let $\bar{G} = G/N$. Prove that if G is generated $x, y \in G$, then $\bar{G}$ is generated by $\bar{x}, \bar{y} \in \bar{G}$.

Solution.

Exercise 2

In the general linear group $GL_3(\mathbb{F})$, consider the subsets

$$H = \begin{bmatrix} 1 & * & * \\ 0 & 1 & * \\ 0 & 0 & 1 \end{bmatrix} \qquad \text{and} \qquad K = \begin{bmatrix} 1 & 0 & * \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

where $*$ represents an arbitrary element of a field $\mathbb{F}$.

(a) Show that H is a subgroup of $GL_3(\mathbb{F})$. *Hint: First, compute the product*

$$\begin{bmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & x & y \\ 0 & 1 & z \\ 0 & 0 & 1 \end{bmatrix}.$$

(b) Show that K is a *normal* subgroup of H.

(c) For $\mathbb{F} = \mathbb{R}$, identify the quotient group H/K (up to isomorphism). *Hint: Let $A, B \in H$. Under what conditions are A and B in the same coset of K? Use this to construct a surjective homomorphism from H.*

Remark. The subgroup H discussed here is called the *Heisenberg group*, and we can actually define it using elements of commutative rings, not just fields. This version of this group with $\mathbb{F} = \mathbb{R}$ was used by Weyl to give an algebraic interpretation of Heisenberg's Uncertainty Principle.

SOLUTION.

Exercise 3

Recall that the Klein four group is $V = \{1, a, b, ab\} = \langle a, b \mid a^2 = b^2 = [a, b] = 1 \rangle \cong C_2 \times C_2$ (see page 47 in the book).

(a) Prove that the subgroup $N = \{e, (1\ 2)(3\ 4), (1\ 3)(2\ 4), (1\ 4)(2\ 3)\}$ in S_4 is isomorphic to the Klein four group.

(b) Prove that N is normal in S_4. *Hint: Use a theorem from lecture on Wednesday; do not use brute force!*

(c) Prove that the subgroup $H = \langle (1\ 2), (3\ 4) \rangle \leq S_4$ is also isomorphic to V, but is not a normal subgroup of S_4.

(d) Identify the quotient group S_4/N by computing the cosets. *Hint: Recall that $|S_4| = 4! = 24$; use the counting formula. Either define an isomorphism between S_4/N and your candidate group, or define a surjection from S_4 to your candidate group. You do not need to show me that your map is a homomorphism; just check for yourself that it really is.*

Inserting images: When computing the cosets, you may write these by hand and insert a clear image of your work, using

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\includegraphics[width=\textwidth]{your-image-name.png}
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(e) How many subgroups are there in S_4 that contain N? (*Do not solve this by brute force!*)

SOLUTION.