Lecture 06

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MAT 150A

Participation Slip

1. Take a slip from the front of the room.
2. Write your full name on the top left corner.
3. You will write down your answer to some clearly marked "Participation Slip" questions during lecture.
4. Hand in your slip at the end of class.

Reminder

- Participation slips won't be graded until Lecture 9.
- From Lecture 9 and onward, your participation slip will be graded for completion.
- A score of 15 (out of 20 lecture days) will receive full credit.

Objects and Morphisms

So far we've been talking about algebraic **objects**: sets, groups, rings, fields, etc.
Now we will study **morphisms** between these objects, i.e. **structure-preserving maps** between them.

Example: Finite-dimensional $\mathbb{R}$ Vector Spaces

- Objects: Finite-dimensional real vector spaces
- Morphisms: Given objects V, W, a morphism from V to W is a *linear map*

$$\phi : V \to W.$$

Linear maps preserve the structure of vector spaces:

- $\phi(\mathbf{0}_V) = \mathbf{0}_W$
- $\phi(v_1 + v_2) = \phi(v_1) + \phi(v_2)$
- $\phi(cv) = c\phi(v)$

Group Homomorphisms

Definition

Let $(S, \square)$ and $(T, \blacktriangle)$ be groups.
A **homomorphism**

$$\varphi : (S, \square) \to (T, \blacktriangle)$$

is a (set) map such that for all $a, b \in S$,

$$\varphi(a \mathbin{\square} b) = \varphi(a) \blacktriangle \varphi(b).$$

Group Homomorphisms

Definition

Let G, G' be groups, written in multiplicative notation.
A **homomorphism**

$$\varphi : G \to G'$$

is a map from G to G' such that for all $a, b \in G$,

$$\varphi(ab) = \varphi(a)\varphi(b).$$

Group Homomorphisms

Examples of homomorphisms

1. det : $GL_n(\mathbb{R}) \to \mathbb{R}^\times$
2. sgn : $S_n \to \{\pm 1\}$
3. exp : $\mathbb{R}^+ \to \mathbb{R}^\times$, where $x \mapsto e^x$
4. $\varphi : \mathbb{Z}^+ \to G$ where $\varphi(n) = a^n$ where a is a fixed element of G
5. $|\cdot| : \mathbb{C}^\times \to \mathbb{R}^\times$

Some important homomorphisms

1. Let G, G' be groups. The **trivial homomorphism** is
$$\begin{aligned} c : G &\to G' \\ g &\mapsto 1_{G'} \end{aligned}$$
2. Let G be a group. The **identity homomorphism** is
$$\begin{aligned} \mathrm{id} : G &\to G \\ g &\mapsto g \end{aligned}$$
3. Let H be a subgroup of G. The **inclusion map** is
$$\begin{aligned} i : H &\hookrightarrow G \\ x &\mapsto x \end{aligned}$$

Group Homomorphisms

Proposition

Let $\varphi : G \to G'$ be a group homomorphism.

1. If $a_1, a_2, \ldots, a_k \in G$, then
$$\varphi(a_1 a_2 \cdots a_k) = \varphi(a_1)\varphi(a_2)\cdots\varphi(a_k).$$
2. $\varphi(1_G) = 1_{G'}$
3. If $a \in G$, then
$$\varphi(a^{-1}) = \varphi(a)^{-1}.$$

Kernel and Image

Let $\varphi : G \to G'$ be a group homomorphism.

Definition

1. The **kernel** of φ is

$$\ker \varphi := \{a \in G \mid \varphi(a) = 1\}.$$

2. The **image** of φ is

$$\operatorname{im} \varphi := \{x \in G' \mid x = \varphi(a) \text{ for some } a \in G\} =: \varphi(G).$$

Participation Slip

1. Prove that $\operatorname{im} \varphi$ is a subgroup of G'.
2. Prove that $\ker \varphi$ is a subgroup of G.

Examples of Kernels

The kernel of the determinant homomorphism

$$\det : GL_n(\mathbb{R}) \to \mathbb{R}^\times$$

is the **special linear group** $SL_n(\mathbb{R})$.

The kernel of the sign homomorphism

$$\mathrm{sgn} : S_n \to \{\pm 1\}$$

is called the **alternating group** A_n, i.e. the subgroup of all *even* permutations.

Cosets

Definition

Let H be a subgroup of G, and let $a \in G$.
Let aH denote the set

$$aH = \{g \in G \mid g = ah \text{ for some } h \in H\}.$$

- This set is called a **left coset** of H in G, because the element a "left-multiplied" with H.
- The set of all left cosets of H in G is

$$\{bH \mid b \in G\}.$$

Warning: The set of left cosets of H is not always a group! We'll come back to this.

Q: How do you eat corn on the cob?

Normal Subgroups

Definition

If $a, g \in G$, the element gag^{-1} is called the **conjugate of** a **by** g**.**

Definition

A subgroup N of G is a **normal subgroup**
if for every $a \in N$, $g \in G$, the conjugate $gag^{-1} \in N$.

If $N \leq G$ is normal, we write $N \trianglelefteq G$.

Example: Subgroups of abelian groups

If G is abelian, then any subgroup H is normal. Why?

When G is nonabelian, not every subgroup is necessarily normal.

Quotient Groups (first pass)

Fact (will prove later)

Let N be a normal subgroup of a group G. Then the set of left cosets of N in G, G/N, forms a group.