Lecture 10

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MAT 150A

Participation Slip

1. Take a slip from the front of the room.
2. Write your full name on the top left corner.
3. You will write down your answer to some clearly marked "Participation Slip" questions during lecture.
4. Hand in your slip at the end of class, in the pile according to the first letter of your surname.

Exam 1

- Wednesday, October 25th, in-class.
- Bring pencil(s) and eraser(s). Scratch paper will be provided with your exam packet.
- As in any proof-based class, you must provide complete proofs for full credit.
  - In particular, your proof must be written in full sentences, and all your variables must be defined.
  - Poor proof-writing practices will incur losses in both correctness and style points.
- See class website for more information, including a mock copy of the front page of the exam.

Exam 1 Advice

We have covered a lot of new concepts and definitions in the past few weeks.

- At a bare minimum, you must know the terminology and definitions.
- You are also expected know the main lemmas, propositions, theorems, etc. that we have proven, especially the ones that have been useful in proving other propositions or in examples.
- Reading proofs may give you a false sense of understanding. Try to prove statements yourself first. Getting stuck is good; you'll identify what you need to review this way.

Recall: Various Facts from Last Time

Counting Formula: For $H \leq G$, $|G| = |H| \cdot [G : H]$.

Corollary

Let $\varphi : G \to G'$ be a homomorphism of finite groups.

- $[G : \ker \varphi] = |\operatorname{im} \varphi|$, and hence $|G| = |\ker \varphi||\operatorname{im} \varphi|$
- $|\ker \varphi|$ divides $|G|$
- $|\operatorname{im} \varphi|$ divides both $|G|$ and $|G'|$.

Lemma 2.6.2

If $\varphi : G \to G'$ is an isomorphism, then the inverse (set) map $\varphi^{-1} : G' \to G$ is also an isomorphism.

Disambiguation: For any $\varphi : G \to G'$, $\varphi^{-1}(g')$ for $g' \in G'$ indicates the **preimage** of g' under φ.

Endomorphisms and Automorphisms

Recall

- An **endomorphism** is a homomorphism G from G to itself.
- An **automorphism** is an <u>isomorphism</u> from G to itself.

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Let G be a group, and fix an element $g \in G$. Prove that "conjugation by g" is an automorphism of G:

$$\begin{aligned} g\bullet : G &\to G \\ a &\mapsto gag^{-1}. \end{aligned}$$

Commutators

Let G be a group.

Definition

For $a, b \in G$, the **commutator** of a and b is

$$[a, b] = aba^{-1}b^{-1}.$$

The following are equivalent:

- $ab = ba$
- $aba^{-1} = b$
- $aba^{-1}b^{-1} = 1$

The elements a and b commute if and only if $[a, b] = 1$.

Commutators

Example

The free group on two elements

$$F_2 = \langle a, b \mid \emptyset \rangle$$

(no relations) is not abelian.

If we **abelianize** the group F_2 by adding the relation $[a, b] = 1$, we get the **direct product**

$$\mathbb{Z} \times \mathbb{Z} \cong \langle a, b \mid [a, b] = 1 \rangle.$$

We can visualize $\mathbb{Z} \times \mathbb{Z}$ as the additive group of integral lattice points in $\mathbb{R}^2$.

We will discuss direct products in detail soon.

Center of a Group

Definition

The **center** of a group G is the subgroup of G consisting of the elements $z \in G$ that commute with every element of the group:

$$Z(G) = \{z \in G \mid [g, z] = 1 \text{ for all } g \in G\}$$

If $z \in Z(G)$, we say that z is **central** in G.

Examples

- The center of $GL_n(\mathbb{R})$ is the set of diagonal matrices.
- The center of S_n is trivial (i.e. $Z(S_n) = \{id\}$) if $n \geq 3$.

Lemma

For any group G, $Z(G) \trianglelefteq G$.

Intersections of subgroups

HW03

If K and H are subgroups of G, then their intersection, $K \cap H$, is also a subgroup of G.

Example

Consider the subgroups $K = 2\mathbb{Z}$ and $H = 3\mathbb{Z}$ inside $G = \mathbb{Z}$. Then $K \cap H = 6\mathbb{Z}$, which is also a subgroup of $\mathbb{Z}$.

Lesson Plans

- We will postpone §2.10 (The Correspondence Theorem) until after Exam 1.
  - You will therefore not be tested on §2.10.
- In this lecture and the next, we'll talk about §2.11 (Direct Products) and §2.12 (Quotient Groups).

Direct Products

Definition

Let A and B be groups. Their **direct product** $A \times B$ is the group with

- underlying set: $A \times B$
- group operation / law of composition:

$$(a_1, b_1)(a_2, b_2) = (a_1 a_2, b_1 b_2).$$

This is as opposed to semi-direct product, or various other types of products, which we will talk about much later in this course.

Characterizing Direct Products

Composition in $A \times B$: $(a_1, b_1)(a_2, b_2) = (a_1 a_2, b_1 b_2)$

The definition of $G \times G'$ seems simple enough, but it is nontrivial to detect whether an arbitrary group has a **product structure**:

$$\begin{tikzcd}
A \arrow[dr, "i_A"] & & A \\
& A \times B \arrow[ur, "p_A"] \arrow[dr, "p_B"] & \\
B \arrow[ur, "i_B"] & & B
\end{tikzcd}$$

- i_A, i_B are **inclusions**: $i_A(a) = (a, 1), i_B(b) = (1, b)$
- p_A, p_B are **projections**: $p_A(a, b) = a$, $p_B(a, b) = b$.