

Lecture 10

Melissa Zhang

MAT 150A

Participation Slip

- ① Take a slip from the front of the room.
- ② Write your full name on the top left corner.
- ③ You will write down your answer to some clearly marked “Participation Slip” questions during lecture.
- ④ Hand in your slip at the end of class, in the pile according to the first letter of your surname.

Exam 1

- Wednesday, October 25th, in-class.
- Bring pencil(s) and eraser(s). Scratch paper will be provided with your exam packet.
- As in any proof-based class, you must provide complete proofs for full credit.
 - In particular, your proof must be written in full sentences, and all your variables must be defined.
 - Poor proof-writing practices will incur losses in both correctness and style points.
- See class website for more information, including a mock copy of the front page of the exam.

We have covered a lot of new concepts and definitions in the past few weeks.

- At a bare minimum, you must know the terminology and definitions.
- You are also expected know the main lemmas, propositions, theorems, etc. that we have proven, especially the ones that have been useful in proving other propositions or in examples.
- Reading proofs may give you a false sense of understanding. Try to prove statements yourself first. Getting stuck is good; you'll identify what you need to review this way.

Recall: Various Facts from Last Time

Counting Formula: For $H \leq G$, $|G| = |H| \cdot [G : H]$.

Corollary

Let $\varphi : G \rightarrow G'$ be a homomorphism of finite groups.

- $[G : \ker \varphi] = |\operatorname{im} \varphi|$, and hence $|G| = |\ker \varphi| |\operatorname{im} \varphi|$
- $|\ker \varphi|$ divides $|G|$
- $|\operatorname{im} \varphi|$ divides both $|G|$ and $|G'|$.

Lemma 2.6.2

If $\varphi : G \rightarrow G'$ is an isomorphism, then the inverse (set) map $\varphi^{-1} : G' \rightarrow G$ is also an isomorphism.

Disambiguation: For any $\varphi : G \rightarrow G'$, $\varphi^{-1}(g')$ for $g' \in G'$ indicates the **preimage** of g' under φ .

Recall

- An **endomorphism** is a homomorphism G from G to itself.
- An **automorphism** is an isomorphism from G to itself.

Participation Slip

Let G be a group, and fix an element $g \in G$. Prove that “conjugation by g ” is an automorphism of G :

$$g \bullet : G \rightarrow G$$
$$a \mapsto gag^{-1}.$$

Let G be a group.

Definition

For $a, b \in G$, the **commutator** of a and b is

$$[a, b] = aba^{-1}b^{-1}.$$

The following are equivalent:

- $ab = ba$
- $aba^{-1} = b$
- $aba^{-1}b^{-1} = 1$

The elements a and b commute if and only if $[a, b] = 1$.

Example

The free group on two elements

$$F_2 = \langle a, b \mid \emptyset \rangle$$

(no relations) is not abelian.

If we **abelianize** the group F_2 by adding the relation $[a, b] = 1$, we get the **direct product**

$$\mathbb{Z} \times \mathbb{Z} \cong \langle a, b \mid [a, b] = 1 \rangle.$$

We can visualize $\mathbb{Z} \times \mathbb{Z}$ as the additive group of integral lattice points in \mathbb{R}^2 .

We will discuss direct products in detail soon.

Center of a Group

Definition

The **center** of a group G is the subgroup of G consisting of the elements $z \in G$ that commute with every element of the group:

$$Z(G) = \{z \in G \mid [g, z] = 1 \text{ for all } g \in G\}$$

If $z \in Z(G)$, we say that z is **central** in G .

Examples

- The center of $GL_n(\mathbb{R})$ is the set of diagonal matrices.
- The center of S_n is trivial (i.e. $Z(S_n) = \{id\}$) if $n \geq 3$.

Lemma

For any group G , $Z(G) \trianglelefteq G$.

HW03

If K and H are subgroups of G , then their intersection, $K \cap H$, is also a subgroup of G .

Example

Consider the subgroups $K = 2\mathbb{Z}$ and $H = 3\mathbb{Z}$ inside $G = \mathbb{Z}$. Then $K \cap H = 6\mathbb{Z}$, which is also a subgroup of \mathbb{Z} .

- We will postpone §2.10 (The Correspondence Theorem) until after Exam 1.
 - You will therefore not be tested on §2.10.
- In this lecture and the next, we'll talk about §2.11 (Direct Products) and §2.12 (Quotient Groups).

Definition

Let A and B be groups. Their **direct product** $A \times B$ is the group with

- underlying set: $A \times B$
- group operation / law of composition:

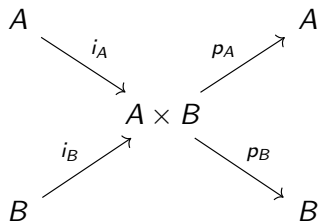
$$(a_1, b_1)(a_2, b_2) = (a_1 a_2, b_1 b_2).$$

This is as opposed to semi-direct product, or various other types of products, which we will talk about much later in this course.

Characterizing Direct Products

$$\text{Composition in } A \times B: (a_1, b_1)(a_2, b_2) = (a_1 a_2, b_1 b_2)$$

The definition of $G \times G'$ seems simple enough, but it is nontrivial to detect whether an arbitrary group has a **product structure**:



- i_A, i_B are **inclusions**: $i_A(a) = (a, 1)$, $i_B(b) = (1, b)$
- p_A, p_B are **projections**: $p_A(a, b) = a$, $p_B(a, b) = b$.