# Lecture 11

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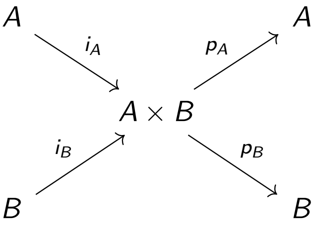
MAT 150A

# Participation Slip

1. Take a slip from the front of the room.
2. Write your full name on the top left corner.
3. You will write down your answer to some clearly marked "Participation Slip" questions during lecture.
4. Hand in your slip at the end of class, in the pile according to the first letter of your surname.

Composition in $A \times B$: $(a_1, b_1)(a_2, b_2) = (a_1 a_2, b_1 b_2)$

The definition of $G \times G'$ seems simple enough, but it is nontrivial to detect whether an arbitrary group has a **product structure**:




- $i_A, i_B$ are **inclusions**: $i_A(a) = (a, 1), i_B(b) = (1, b)$
- $p_A, p_B$ are **projections**: $p_A(a, b) = a$, $p_B(a, b) = b$.

# Direct Products

**Example**

Given $C_6 = \langle x \mid x^6 = 1 \rangle$, it is not obvious that $C_6 \cong C_2 \times C_3$. To prove this, we need to either

1. exhibit the product structure, i.e. check that we have the decomposition in the diagram
2. if we have a guess of what the factor groups (subgroups) are, we can exhibit an explicit isomorphism.

**Proposition 2.11.3**

Let $r$ and $s$ be relatively prime integers. A cyclic group of order $rs$ is isomorphic to the (direct) product of a cyclic group of order $r$ and a cyclic group of order $s$.

This is **not** true when $r$ and $s$ are **not** relatively prime.

# Describing Product Groups

Let $H, K \leq G$, and let $f : H \times K \to G$ be the multiplication map in $G$, i.e. $f(h, k) = hk$. The image of $f$ is the set

$$HK = \{hk \mid h \in H, k \in K\}.$$

**Proposition 2.11.4**

(a) $f$ is injective if and only if $H \cap K = \{1\}$.

(b) $f$ is a homomorphism $H \times K \to G$, if and only if elements of $H$ commute with those of $K$: $hk = kh$ for all $h \in H, k \in K$.

(c) If $H \trianglelefteq G$, then $HK \leq G$.

(d) $f$ is an isomorphism $H \times K \to G$ if and only if $H \cap K = \{1\}$, $HK = G$, and both $H, K \trianglelefteq G$.

$f : H \times K \to G$, $(h, k) \mapsto hk$.

(a) $f$ is injective if and only if $H \cap K = \{1\}$.

**Proof.**

- ($\Rightarrow$, by contrapositive) If $H \cap K$ contains some $x \neq 1$, then $x^{-1}H$ as well, so $f(x^{-1}, x) = 1 = f(1, 1)$, so $f$ is not injective.
- ($\Leftarrow$) Suppose $H \cap K = \{1\}$. Let $(h_1, k_1), (h_2, k_2) \in H \times K$ such that $h_1 k_1 = h_2 k_2$.
  By multiplying on the left by $h_1^{-1}$, then on the right by $k_2^{-1}$, we have $k_1 k_2^{-1} = h_1^{-1} h_2 = 1$.
  Hence $h_1 = h_2$ and $k_1 = k_2$, so $f$ is injective.

$f : H \times K \to G$, $(h, k) \mapsto hk$.

(b) $f$ is a homomorphism $H \times K \to G$, if and only if elements of $H$ commute with those of $K$: $hk = kh$ for all $h \in H, k \in K$.

**Proof.**

- Let $(h_1, k_1), (h_2, k_2) \in H \times K$.
- Firstly, $f((h_1, k_1)(h_2, k_2)) = f((h_1 h_2, k_1 k_2)) = h_1 h_2 k_1 k_2$.
- On the other hand, $f(h_1, k_1) f(h_2 k_2) = (h_1 k_1)(h_2 k_2) = h_1 k_1 h_2 k_2$.
- $f$ is a homomorphism iff $h_1 h_2 k_1 k_2 = h_1 k_1 h_2 k_2$. By cancellation, this is true iff $h_2 k_1 = k_1 h_2$.
- ($k_1$ and $h_2$ were arbitrary, so we have shown that $hk = kh$ for all $h \in H, k \in K$.)

$f : H \times K \to G$, $(h, k) \mapsto hk$.

(c) If $H \trianglelefteq G$, then $HK \leq G$.

**Proof.**

- Suppose $H \trianglelefteq G$.
- (Identity) Since $1 \in H$ and $1 \in K$, $1 \in HK$.
- (Closure) Since $H \trianglelefteq G$, for any $k \in K \subset G$, $khk^{-1} = h' \in H$, i.e. $kh = h'k$. Thus $kH = Hk$, and therefore $KH = HK$. Then $(HK)(HK) = HKHK = HHKK = HK$, so $HK$ is closed under multiplication (i.e. the group operation).
- (Inverse) Let $hk \in HK$. Then $(hk)^{-1} = k^{-1}h^{-1} \in KH = HK$ (see above), so $HK$ is closed under inverses.

$f : H \times K \to G$, $(h, k) \mapsto hk$.

(d) $f$ is an isomorphism $H \times K \to G$ if and only if (i) $H \cap K = \{1\}$, (ii) $HK = G$, and (iii) both $H, K \trianglelefteq G$.

**Proof.**

($\Leftarrow$) Suppose $H$ and $K$ satisfy the conditions (i), (ii), (iii).

- Since $f$ is injective (i) and surjective (ii), $f$ is bijective.
- It remains to show that $f$ is a homomorphism; to show this, we use part (b): we just need to show that $hk = kh$ for all $h \in H, k \in K$.
  - Since $K \trianglelefteq G$, $[h, k] = (hkh^{-1})k^{-1} \in K$.
  - Since $H \trianglelefteq G$, $[h, k] = h(kh^{-1}k^{-1}) \in H$.
  - Since $H \cap K = \{1\}$, $[h, k] = 1$, i.e. $hk = kh$.

($\Rightarrow$) Suppose $f$ is an isomorphism. Verify the statements (i)–(iii) on the isomorphic group $H \times K$. ■

## Proposition 2.11.4 (d)

$f : H \times K \to G$, $(h, k) \mapsto hk$, is an isomorphism if and only if $H \cap K = \{1\}$, $HK = G$, and both $H, K \trianglelefteq G$.

## Example: $\mathbb{Z} \times \mathbb{Z} \cong \langle a, b \mid [a, b] = 1 \rangle$

- Let $H = \langle a \rangle$ and $K = \langle b \rangle$.
  Note that $\mathbb{Z} \cong H \cong K$: $n \mapsto a^n$, $n \mapsto b^n$.
- There is no relation of the form $a^k = b^j$ (or any combination of relations that implies this); thus $H \cap K = \{1\}$.
- To see that $HK = G$, we just need to show that $G \subseteq HK$.
  Any $g \in G$ is a word in $a$ and $b$ (and $a^{-1}, b^{-1}$). Since $[a, b] = 1$, we can commute all the powers of $a$ to the left. Thus $g = a^k b^j$ for some $k, j \in \mathbb{Z}$, and is therefore in $HK$.
- Since $G$ is abelian, both $H, K \trianglelefteq G$. ■

# Practice

**Exercise**

Classify the groups of order 4.