Lecture 12 Solutions

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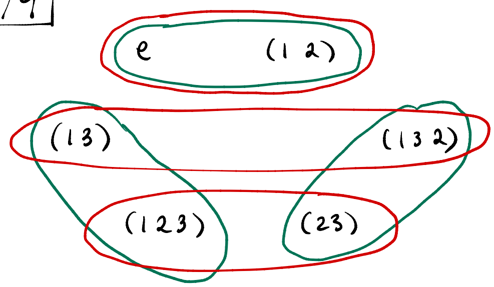

left cosets: $(23)\circ(12) = (132)$

right cosets: $(12)\circ(23) = (123)$

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prop. Let $H \leq G$, $g \in G$. Then $|gH| = |Hg|$. (For a set S, let $|S| = \text{card}(S)$.)

Proof 1 We know $|H| = |gH|$. Using the same proof, we show $|H| = |Hg|$:

The set map $(\cdot g): H \to Hg$ has inverse $(\cdot g^{-1}): Hg \to H$ and is therefore a bijection.

Thus $|gH| = |H| = |Hg|$.

Proof 2 The set map $g^{-1}\cdot(-)\cdot g : gH \to Hg$ has inverse $g\cdot(-)\cdot g^{-1}$, and is therefore a bijection.

prop. # left cosets of H in G = # right cosets of H in G.

Pf. Let $g\bullet : G \to G$ denote the "conjugation by g" automorphism.

Define a set map $\varphi: \{\text{left cosets}\} \to \{\text{right cosets}\}$ induced by the automorphisms $g^{-1}\bullet$.

This is well-defined:

Let $gh_1, gh_2 \in gH$. Then $g^{-1}\bullet(gh_1) = g^{-1}(gh_1)g = h_1 g \in Hg$
and $g^{-1}\bullet(gh_2) = g^{-1}(gh_2)g = h_2 g \in Hg$.

The inverse function φ^{-1} is similarly defined by the maps $g\bullet$.

Therefore φ is a set bijection.

Prop 2.8.17 Let $H \leq G$. TFAE:

① $H \trianglelefteq G$

② $\forall g \in G$, $gHg^{-1} = H$

③ $\forall g \in G$, $gH = Hg$

④ every left coset of H is a right coset.

Pf.

①⇒② Let $ghg^{-1} \in gHg^{-1}$. Since $H \trianglelefteq G$, $ghg^{-1} \in H$ by definition.
Therefore $gHg^{-1} \subseteq H$.

Now let $h \in H$. Since $H \trianglelefteq G$, $g^{-1}hg \in H$. Then $h = g(g^{-1}hg)g^{-1} \in gHg^{-1}$.
So $H \subseteq gHg^{-1}$. By double inclusion, $gHg^{-1} = H$.

②⇒③ If $gHg^{-1} = H$, then by right-multiplying both sets by g,
we immediately have $gH = Hg$.

③⇒① Suppose $\forall g \in G$, $gH = Hg$. For any $g \in G$ and $h \in G$, $\exists h'$ s.t. $gh = h'g$.
Then $ghg^{-1} = h'gg^{-1} = h' \in H$, so $H \trianglelefteq G$.

③⇔④ The ⇒ direction is immediate.

Assume every left coset is a right coset, ie for any $g \in G$, $\exists g' \in G$ such that $gH = Hg'$.

Since $1 \in H$, $g' \in Hg$, so $Hg \cap Hg' \neq \emptyset$. Since the right cosets partition G, we must have $Hg = Hg'$. Therefore $gH = Hg' = Hg$.

(a) Recall that conjugation by g is an automorphism of G.

Therefore the restriction
$$\begin{aligned} \varphi : H &\longrightarrow G \\ h &\longmapsto ghg^{-1} \end{aligned}$$
is also a homomorphism. Since $gHg^{-1} = \operatorname{im}\varphi$, it is a subgroup of G.

(b) Since $gHg^{-1} \leq G$, and conjugation by g is an automorphism, it gives an isomorphism $g\cdot(-)\cdot g^{-1} : H \longrightarrow gHg^{-1} \leq G$.

Then $|gHg^{-1}| = r$, so $gHg^{-1} = H$ (since H is the only subgroup of G of order r). By Prop. 2.8.17, $H \trianglelefteq G$.