Lecture 16

① Useful facts / calculations: $A = \begin{pmatrix} a & b \\ 0 & d \end{pmatrix}$ $X = \begin{pmatrix} x & y \\ 0 & z \end{pmatrix}$

- $A^{-1} = \begin{pmatrix} a & b \\ 0 & d \end{pmatrix}^{-1} = a^{-1}d^{-1}\begin{pmatrix} d & -b \\ 0 & a \end{pmatrix}$

- $AX = \begin{pmatrix} a & b \\ 0 & d \end{pmatrix}\begin{pmatrix} x & y \\ 0 & z \end{pmatrix} = \begin{pmatrix} ax & ay+bz \\ & dz \end{pmatrix}$

- $AXA^{-1} = \frac{1}{ad}\begin{pmatrix} a & b \\ 0 & d \end{pmatrix}\begin{pmatrix} x & y \\ 0 & z \end{pmatrix}\begin{pmatrix} d & -b \\ 0 & a \end{pmatrix} = \frac{1}{ad}\begin{pmatrix} ax & ay+bz \\ & dz \end{pmatrix}\begin{pmatrix} d & -b \\ 0 & a \end{pmatrix}$

$= \frac{1}{ad}\begin{pmatrix} axd & -axb + a^2y + abz \\ & dza \end{pmatrix} = \begin{pmatrix} x & -\frac{xb}{d} + \frac{ay}{d} + \frac{bz}{d} \\ 0 & z \end{pmatrix}$

$= \begin{pmatrix} x & \frac{1}{d}(-bx + bz + ay) \\ 0 & z \end{pmatrix} =: \begin{pmatrix} x & y' \\ 0 & z \end{pmatrix}$

(a) $y=0$: $S = \left\{\begin{pmatrix} x & 0 \\ 0 & z \end{pmatrix}\right\}$

- $=$ diagonal matrices in $GL_n(\mathbb{F}) \Rightarrow S \leq G$
- If $X = \begin{pmatrix} x & 0 \\ 0 & z \end{pmatrix}$ then $y' = \frac{1}{d}(-bx + bz)$ is not 0 in general.

$\Rightarrow S \ntrianglelefteq G$

(b) $z=1$: $S = \left\{\begin{pmatrix} x & y \\ 0 & 1 \end{pmatrix}\right\}$

- $S \leq G$: $I_2 \in S$, $\frac{1}{x}\begin{pmatrix} 1 & -y \\ 0 & x \end{pmatrix} \in S$, $\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x & y \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} ax & ay+b \\ 0 & 1 \end{pmatrix} \in S$.
- If $X = \begin{pmatrix} x & y \\ 0 & 1 \end{pmatrix}$ then $AXA^{-1} = \begin{pmatrix} x & \frac{1}{d}(-bx + bz + ay) \\ 0 & 1 \end{pmatrix} \in S$

$\Rightarrow S \trianglelefteq G$.

- Define $\varphi: G \longrightarrow \mathbb{F}^\times$ by $\begin{pmatrix} x & y \\ 0 & z \end{pmatrix} \mapsto z$

$\varphi(A)\varphi(X) = dz$ & $\varphi(AX) = dz \Rightarrow \varphi$ is a hom., also clearly surjective: $\begin{pmatrix} * & * \\ 0 & z \end{pmatrix} \mapsto z$.

$\ker \varphi = \left\{ \begin{pmatrix} x & y \\ 0 & z \end{pmatrix} \;\middle|\; z = 1 \right\} = S.$

By 1^{st} isom. theorem, $G/S \cong \mathbb{F}^\times$.

(c) $x = z$ $\quad S = \left\{ \begin{pmatrix} x & y \\ 0 & x \end{pmatrix} \right\}$

- $I_2 \in S$, $\frac{1}{x^2}\begin{pmatrix} x & -y \\ 0 & x \end{pmatrix} \in S$, $\begin{pmatrix} a & b \\ 0 & a \end{pmatrix}\begin{pmatrix} x & y \\ 0 & x \end{pmatrix} = \begin{pmatrix} ax & ay + bx \\ & ax \end{pmatrix} \in S. \Rightarrow S \leq G.$
- If $X = \begin{pmatrix} x & y \\ 0 & x \end{pmatrix}$, then $AXA^{-1} = \begin{pmatrix} x & * \\ 0 & x \end{pmatrix}$ something $\in S \Rightarrow S \trianglelefteq G.$

If it's unclear how we should define φ, try to understand the cosets, ie by understanding the equivalence relation:

If X, A are in the same coset of S, then $XA^{-1} \in S$.

$$XA^{-1} = \frac{1}{ad}\begin{pmatrix} x & y \\ 0 & z \end{pmatrix}\begin{pmatrix} d & -b \\ 0 & a \end{pmatrix} = \frac{1}{ad}\begin{pmatrix} xd & -xb + ya \\ & za \end{pmatrix}$$

$= \begin{pmatrix} x/a & * \\ 0 & z/d \end{pmatrix} \in S$ iff $\frac{x}{a} = \frac{z}{d}$ ie. $xd = az$. ie $\frac{x}{z} = \frac{a}{d}$.

Therefore each coset of S corresponds to a choice of one element of the field: $\frac{a}{d}$.

Now use 1^{st} isom thm to prove your guess:

Define a surjective map $\varphi : G \twoheadrightarrow \mathbb{F}$
$$\begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \mapsto \frac{a}{d}$$

$$\begin{tikzcd} G \arrow[r, two heads, "\varphi"] \arrow[d, "\pi"'] & \mathbb{F}^\times \\ G/S & \end{tikzcd}$$

- surjective: let $d = 1$, $a \in \mathbb{F}$
- $\ker \varphi = \left\{ \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} : \frac{a}{d} = 1 \right\}$ ie when $a = d$.

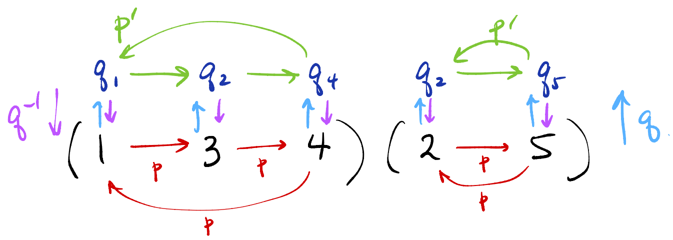
So $G/S \cong \mathbb{F}^\times$ by the 1^{st} isom theorem.

② $p = (134)(25)$ $\quad q = (1452)$ $\quad q^{-1} = (2541)$

$$qpq^{-1} = \big(q(1)\ q(3)\ q(4)\big)\big(q(2)\ q(5)\big)$$

$$p = (1\ 3\ 4)(2\ 5)$$
$$p' = (4\ 3\ 5)(1\ 2)$$

Why does this work? Say $q(i) = q_i \in \{1,2,3,4,5\} = [5]$



10 AM class: We will go over this on Friday.