Lecture 16

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MAT 150A

Don't forget to pick up a participation slip!

Today

The goal is to help you do HW05, which is very instructive.

1. quotient groups example (§2.12)
2. conjugation classes in S_n (§7.5)
3. the free group, generators and relations (§7.9, 7.10)

I will post my personal lecture notes on the class website this evening.

Quotient Groups Example

Example

Let G be the subgroup of upper triangular matrices $\begin{bmatrix} x & y \\ 0 & z \end{bmatrix} \in GL_n(\mathbb{R})$. We can also write this set as $\begin{bmatrix} * & * \\ 0 & * \end{bmatrix}$.

For each of the following subsets, determine whether S is a subgroup and whether $S \trianglelefteq G$. If $S \trianglelefteq G$, identify the quotient group G/S.

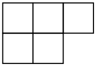
a. Participation Slip S is the subset defined by $y = 0$.
b. Participation Slip S is the subset defined by $z = 1$.
c. S is the subset defined by $x = z$.

Hint: Compute an arbitrary conjugation AXA^{-1} first, and then answer the questions.

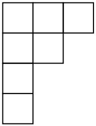
Cycle Type of Permutations in S_n

Let p a permutation in S_n. The **cycle type** of p describes how the cycle notation for p partitions the set $[n]$.

- For $p = (1\ 3\ 4)(2\ 5) \in S_5$, the cycle type of p is "3,2".
  Sometimes written $3 + 2$, or $2 + 3$.



- For $p = (1\ 3\ 4)(2\ 5) \in S_7$, the cycle type of p is "3,2,1,1".



We usually write the block sizes in decreasing order, as a convention.

Conjugation in S_n

Recall our convention for composing permutations in this class:
Let $p = (1\ 3\ 4)(2\ 5)$, $q = (1\ 4\ 5\ 2)$. Then $q^{-1} = (2\ 5\ 4\ 1)$.

start	1 2 3 4 5
apply q^{-1}	2 5 3 1 4
apply p	5 2 4 3 1
apply q	2 1 5 3 4

i.e. qpq^{-1} is the permutation function $[5] \to [5]$ given by the following chart:

i	1 2 3 4 5
$qpq^{-1}(i)$	2 1 5 3 4

So the cycle notation for qpq^{-1} is $(1\ 2)(3\ 5\ 4)$.

Conjugation in S_n

Proposition

Conjugation in S_n preserves **cycle type**.

Algorithm (§7.5 in the text)

1. Map letters back to indices by using q^{-1}.
2. Permute the indices by p.
3. Map indices back to letters using q.

This algorithm shows that the cycle type does not change.

Generators and Relations

In the next two weeks, we'll be focusing on **group actions** and **symmetries**, and will make use of the generators and relations method of describing reasonably small groups.

The Free Group

The free group on n letters $S = \{s_1, s_2, \ldots, s_n\}$, denoted F_n or F_S, is the group consisting of all finite-length words in the generators s_i and their inverses.
(The multiplication operation is concatenation.)

Generators and Relations

Definition

Let G be a group. A **(finite) group presentation** $\langle\ S \mid R\ \rangle$ of G is the data consisting of

- finitely many generators $S = \{g_1, g_2, \ldots, g_n\} \subset G$ ("letters") and
- finitely many relations $R = \{r_1, r_2, \ldots, r_k\}$ (which are themselves "words" in the letters)

such that $G \cong F_S/N$, for $N =$ the smallest normal subgroup of F_S containing the set R.

We can actually make this definition more generally, for sets S and R that are not finite, but those are less useful. The groups that have finite group presentations are called **finitely presented** or **finitely presentable**.