Lecture 18

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MAT 150A

Don't forget to pick up a participation slip!

Reminders

- There is a class calendar available on our class website.
- Exam 2 is next Wednesday. This a cumulative exam. I will post a study guide with practice problems later this week.
- HW06 will be due after Exam 2; no homework due next week.

The Orthogonal Group $O(2)$

Recall: The group $O(2) = O(2, \mathbb{R})$ is the subgroup of $GL(2, \mathbb{R})$ consisting of matrices with orthonormal columns:

$$O(2) = \left\{ \begin{bmatrix} \mathbf{p_1} & \mathbf{p_2} \end{bmatrix} \mid \mathbf{p_i} \cdot \mathbf{p_j} = \delta_{ij} \right\}$$

- Here, δ_{ij} is the **Kronecker delta function**, given by

$$\delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{otherwise.} \end{cases}$$

Bonus: Semi-Direct Products (first pass)

$O(2)$ is a **semi-direct product** of S^1 (rotation) and $\mathbb{Z}/2\mathbb{Z}$ (reflection):

$$O(2) = S^1 \rtimes \mathbb{Z}/2\mathbb{Z}.$$

- The underlying set is still the Cartesian product of S^1 and $\mathbb{Z}/2\mathbb{Z}$.
- But multiplication is slightly different from multiplication in the direct product: you commute an element $t \in \mathbb{Z}/2\mathbb{Z}$ past a rotation $\rho \in S_1$ at the cost of conjugating ρ by t:

$$\begin{aligned}(\rho_1, t_1) \cdot (\rho_2, t_2) \quad &\rightsquigarrow \quad \rho_1 t_1 \rho_2 t_2 \\ &\rightsquigarrow \quad \rho_1 \varphi_{t_1}(\rho_2) t_1 t_2 \quad \rightsquigarrow \quad (\rho_1 \varphi_{t_1}(\rho_2), t_1 t_2)\end{aligned}$$

This topic is not covered in Artin, so we will not say much more about it. But semi-direct products show up all the time, and it's helpful to know about them.

Conjugation ~ Change of Basis ~ Change of Perspective

Recall: Let A be a matrix in the "old" basis, and A' the matrix in the "new" basis. Then there is a change-of-basis matrix P such that

$$A' = P^{-1}AP$$

Participation Slip

Consider the matrix $A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$ and the change-of-basis matrix P corresponding to $\rho_{\pi/2}$.

1. Sketch the unit circle $S \subset \mathbb{R}^2$.
2. Sketch the image of S under A.
3. Sketch the image of S under $P^{-1}AP$.
4. Compare your answers for (b) and (c), and explain why I might have used the term change of perspective.

The Mirror World

Let τ denote reflection across the e_1-axis. ($= r$ in the book)

Observation / Simple Computation

For any rotation $\rho = \rho_\theta$, we have $\tau\rho\tau = \rho^{-1}$:

$$\tau\rho = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} c & -s \\ s & c \end{bmatrix} = \begin{bmatrix} c & s \\ -s & c \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \rho^{-1}\tau$$

Why does a mirror reverse left and right and not top and bottom?

The Mirror World

Why does a mirror reverse left and right and not top and bottom?

$$\tau\rho = \rho^{-1}\tau$$

In this physical example,

- $\tau =$ reflect across the plane of the mirror
  $=$ transfer your consciousness to your mirror self
- $\rho =$ rotation, e.g. by 45° CCW

Finite Subgroups of $O(2)$

We will prove this theorem:

Theorem 6.4.1

Let G be a finite subgroup of the orthogonal group $O(2) = O(2, \mathbb{R}) = O_2$ in the text. There is an integer n such that G is one of the following groups:

1. C_n: the cyclic group of order n generated by the rotation ρ_θ, where $\theta = 2\pi/n$.
2. D_n: the dihedral group of order $2n$ generated by ρ_θ and a reflection r' about a line ℓ through the origin.

Why are translations not relevant here?

Discrete Subgroups of $\mathbb{R}^+$

Definition

A subgroup Γ of the additive group $\mathbb{R}^+$ is called **discrete** if there exists some $\varepsilon > 0$ such that for all nonzero $c \in \Gamma$, $|c| \geq \varepsilon$.

If a set of points is *discrete*, you should think of them as isolated, i.e. if you zoom in enough, then only one point will show up on your screen at any one time.

Lemma 6.4.6

Let Γ be a discrete subgroup of $\mathbb{R}^+$. Then either $\Gamma = \{0\}$, or Γ is the set $a\mathbb{Z}$ of integer multiples of some positive real number $a \in \mathbb{R}$.

The proof is analogous to the proof that subgroups of $\mathbb{Z}$ are of the form $\{0\}$ or $a\mathbb{Z}$. Note that a can be irrational.

Lemma 6.4.6

Let Γ be a discrete subgroup of $\mathbb{R}^+$. Then either $\Gamma = \{0\}$, or Γ is the set $a\mathbb{Z}$ of integer multiples of some positive real number $a \in \mathbb{R}$.

Proof. If $a, b \in \Gamma$ and $a \neq b$, then $|a - b| \geq \varepsilon$ (since $a - b \in \Gamma$).

- Suppose $\Gamma \neq \{0\}$. WTS $\Gamma = a\mathbb{Z}$ for some $a > 0$.
  - Then there exists a nonzero element $b \in \Gamma$, as well as its inverse $-b \neq 0$. So Γ contains a positive element a'.
  - Any bounded interval contains finitely many elements of Γ.
  - Choose the smallest positive element a in the bounded interval $[0, a']$. Then a is also the smallest positive element of Γ.
- We now show $\Gamma = a\mathbb{Z}$.
  - $a \in \Gamma$, so $a\mathbb{Z} \leq \Gamma$.
  - Let $b \in \Gamma$; then $b = ra$ for some $r \in \mathbb{R}$.
  - Write $r = m + r_0$, where $m \in \mathbb{Z}$, $r_0 \in [0, 1)$.
  - Since Γ is a group, $b' = b - ma \in \Gamma$, and $b' = r_0 a$.
  - So $0 \leq b' < a$. By minimality of a, $b' = 0$.
  - Hence $b = ma \in a\mathbb{Z}$, so $\Gamma \subset a\mathbb{Z}$, so $\Gamma = a\mathbb{Z}$. □

Finite Subgroups of $O(2)$

Theorem 6.4.1

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1. C_n: the cyclic group of order n generated by the rotation ρ_θ, where $\theta = 2\pi/n$.
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Finite Subgroups of $O(2)$

Theorem 6.4.1, abridged

If G is a finite subgroup of $O(2)$, then it is either a C_n or D_n.

Proof. Recall that $O(2) = S^1 \rtimes \mathbb{Z}/2\mathbb{Z}$, and every element is of the form $\rho_\theta\tau$, where τ is reflection across the e_1-axis.

Case 1: All $g \in G$ are rotations. It suffices to prove that G is cyclic.

- Let $\Gamma = \{\alpha \in \mathbb{R} \mid \rho_\alpha \in G\}$.
- Then $\Gamma \in \mathbb{R}^+$, and $2\pi \in \Gamma$. Since G is finite, Γ is discrete, so $\Gamma = \alpha\mathbb{Z}$ for some $\alpha \in \mathbb{R}$.
- Then G consists of rotations through integer multiple of α, and there is some n such that $n\alpha = 2\pi$ (i.e. $\alpha = 2\pi/n$).
- So $G \cong C_n$.

Case 2: G contains a reflection r'.

- By a **change of coordinates** (i.e. change of basis), we may assume $\tau \in G$.
  - In other words, perform a change of basis such that r' is taken to τ, by conjugating everything in G to an isomorphic subgroup in $O(2)$.
- Let $H \leq G$ denote the subgroup consisting of rotations that are elements of G.
  - By Case 1, H is cyclic, and is generated by some ρ_θ, for some $\theta = 2\pi/n$.
  - Then the $2n$ products ρ_θ^k and $\rho_\theta^k \tau$, for $0 \leq k \leq n-1$ are in G, so $D_n \leq G$.
- To show $D_n = G$, it remains to show that any $g \in G$ is of this form.
  - If g is a rotation, then $g \in H$ already.
  - If g is a reflection, write it as $\rho_\alpha r$ for some $\rho_\alpha \in O(2)$. But since g and r are both in G, we also have $\rho_\alpha \in G$.
  - So $g \in D_n$. □