Lecture 21

$$\mathrm{Isom}(\mathbb{R}^2) = T \rtimes O(2) = \underset{t_a}{T} \rtimes (\underset{\rho_\theta}{S^1} \rtimes \underset{s}{\mathbb{Z}/2\mathbb{Z}})$$

$s \in \{1, \tau\}$ where τ = reflection across the e_1-axis

$\Rightarrow$ any element can be written as $t_a \rho_\theta s$

If $G \leq \mathrm{Isom}(\mathbb{R}^2)$ is discrete, $G = \underset{\uparrow \text{ translations}}{L} \rtimes \bar{G} \leftarrow$ point group

Q1 How do we write $\tau \rho_\alpha$ in the form $t_a \rho_\theta s$?

Recall In $O(2) = S^1 \rtimes \mathbb{Z}/2\mathbb{Z}$: discrete $G \leq O(2) \Rightarrow G \cong C_n$ or D_n

$(\rho_\theta \tau)^2 = 1 \quad \Rightarrow \rho_\theta \tau = (\rho_\theta \tau)^{-1} = \tau \rho_{-\theta}$

So if we see $\tau \rho_\alpha$, we can rewrite it as $\rho_{-\alpha} \tau$.

Q2 If $\bar{g} \in O(2)$, how do we write $\bar{g} t_v$ in the form above?

Let $\bar{g} = \rho_\alpha s$. Then $\bar{g} t_v = \underbrace{\bar{g} t_v \bar{g}^{-1}}_{\substack{\text{action of } O(2) \\ \text{on } t_v \text{ — conjugation}}} \bar{g}$

$O(2) \curvearrowright T$: Consider $\bar{g} t_v \bar{g}^{-1} = \rho_\theta s t_v s \rho_{-\theta}$.

Consider 2 cases separately: let $v = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$

① $\tau t_v \tau = t_{\begin{bmatrix} v_1 \\ -v_2 \end{bmatrix}} = t_{\tau(v)} \quad \Rightarrow \tau t_v = t_{\tau(v)} \tau$

$$\begin{bmatrix} x \\ y \end{bmatrix} \xmapsto{\tau} \begin{bmatrix} x \\ -y \end{bmatrix} \xmapsto{t_v} \begin{bmatrix} x+v_1 \\ -y+v_2 \end{bmatrix} \xmapsto{\tau} \begin{bmatrix} x+v_1 \\ y-v_2 \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} v_1 \\ -v_2 \end{bmatrix}$$

② $\rho_\alpha t_w \rho_{-\alpha} = t_{\rho_\alpha(w)} \quad \Rightarrow \rho_\alpha t_w = t_{\rho_\alpha(w)} \rho_{-\alpha}$.

$$e^{-i\alpha} t_w e^{-i\alpha}(z) = e^{i\alpha} t_w (e^{-i\alpha} z) = e^{i\alpha}(e^{-i\alpha} z + w) = e^{i\alpha} e^{-i\alpha} z + e^{i\alpha} w = z + \overbrace{e^{i\alpha} w}^{\rho_\alpha(w)}$$

$\Rightarrow (\rho_\alpha s) t_v (s \rho_{-\alpha}) = \rho_\alpha (t_{s(v)}) \rho_{-\alpha} = t_{\rho_\alpha s(v)}$.

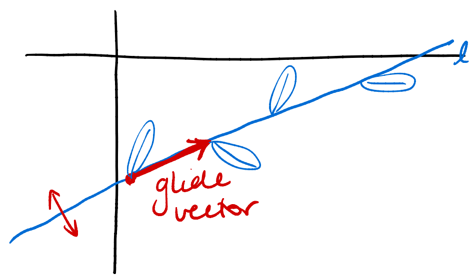
Prop. Let $\bar{g} \in O(2)$. If $t_a \in T$, then $t_{\bar{g}(a)} = \bar{g}\, t_a\, \bar{g}$.

For a discrete $G \leq \mathrm{Isom}(\mathbb{R}^2)$ $(= L \rtimes \bar{G})$,

if $t_a \in L$, then $t_{\bar{g}(a)} \in L$ as well.

Demonstration: Glide reflections g and glide vectors

Let $g = t_a \rho_\alpha \tau \in \mathrm{Isom}(\mathbb{R}^2)$ be a glide reflection.



Participation slip:

① What is the angle of the line of reflection makes w/ the e_1-axis?

$\bar{g} = \rho_\alpha \tau$; the line of reflection makes angle $\alpha/2$ with e_1-axis.

② What is the glide vector v? i.e. we reflect across line ℓ and then glide by this vector v.

Observe: $g = t_a \rho_\alpha \tau$. Then g^2 is just translation by $2v$.

So compute g^2:

$$g^2 = t_a \rho_\alpha \tau\, t_a \rho_\alpha \tau = t_a (\rho_\alpha \tau) t_a \rho_\alpha \tau = t_a\, t_{\rho_\alpha \tau(a)} \underbrace{\rho_\alpha \tau \rho_\alpha \tau}_{=1}$$

$$= t_{a + \rho_\alpha \tau(a)}$$

So the glide vector of g is $\frac{1}{2}(a + \rho_\alpha \tau(a))$.

thm 6.5.12 Crystallographic Restriction)

Let Λ be a discrete nontrivial subgroup of $\mathbb{R}^2$ $(\Lambda \neq \{\vec{0}\})$

and let $Sym(\Lambda)$ denote the group of symmetries of Λ.

Let $H \subset O(2)$ be a subgroup of $Sym(\Lambda)$. ie $H < O(2) \cap Sym(\Lambda)$.

Then ① every rotation ρ in H has order $\in \{1,2,3,4,6\}$

and furthermore, ② $H \cong C_n$ or D_n, for $n \in \{1,2,3,4,6\}$.

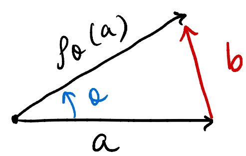
Pf. It suffices to prove (a).

Let ρ_θ be a rotation in H.

Let $a \in \Lambda$ be a minimal length translation vector: $t_a \in Sym(\Lambda)$

Then $\rho_\theta t_a = t_{\rho_\theta(a)} \in Sym(\Lambda) \Rightarrow \rho(a) \in \Lambda$.

Let $b = \rho(a) - a$.



From the figure, we see $\|b\| < \|a\|$ if $\theta < \pi/3$, so by minimality of a,

$\theta \geq \pi/3$. $\Rightarrow ord(\rho_\theta) \leq 6$.

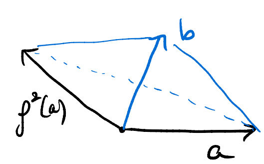
We can easily construct lattices Λ w/ symmetries ρ_θ of order

$n = 1, 2, 3, 4, 6$.

Finally, to show $\theta = 2\pi/5$ does not occur: if $\rho_\theta \in H$, then

$b = \rho_\theta^2 a + a \in \Lambda$ as well. But then b is shorter than a, which

contradicts the minimality of a:



$\boxtimes$

§6.6: We will not cover. Basically:

If $\Lambda \leq \mathbb{R}^2$ is a lattice, ie $\Lambda \cong \mathbb{Z}a \oplus \mathbb{Z}b$,

then $G = \mathrm{Sym}(\Lambda)$ is a plane (2D-) crystallographic group

or "wallpaper groups".

§6.6 Classifies these: there are exactly 17 isomorphism classes.