LECTURE 22

HW 07 will be short, based on material from today's lecture.

Group Actions

Groups can act on more than geometric objects:

① $\tau: \mathbb{C} \longrightarrow \overline{\mathbb{C}}$, $z \mapsto \bar{z}$ — complex conjugation is an <u>automorphism</u> of the field $\mathbb{C}$

② Similarly, conjugation in $\mathbb{F} = \mathbb{Q}[\sqrt{2}]$: $a + b\sqrt{2} \longmapsto a - b\sqrt{2}$

We usually use the term "<u>automorphism</u>" to describe a "symmetry" of an algebraic object. More generally:

<u>defn.</u> (Group action) An operation of a group G on a set S is an assignment
$$G \times S \longrightarrow S$$
$$(g, s) \longmapsto g * s$$

Often denote by $G \circlearrowright S$

where
- $1 * s = s \quad \forall s \in S$
- the action is associative: $(gg') * s = g * (g' * s)$

The action of a particular element can be written $\varphi_g : S \to S$, $s \mapsto gs$.

<u>Q.</u> Why is φ_g a bijective map?

Orbits & Stabilizers

Let $G \curvearrowright S$. Eg. Imagine $\mathbb{Z}/3\mathbb{Z} \curvearrowright$ globe (sphere)

or $S^1 \curvearrowright$ globe (sphere)

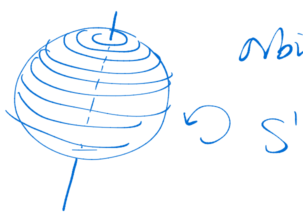
<u>defn</u>. Fix an element $s \in S$. The <u>orbit</u> of s is

$$O_s = \{s' \in S \mid s' = gs \text{ for some } g \in G\} \subseteq S$$

ie O_s = all pts in S that are related to s by the action of G. The group acts independently on each orbit.

Claim This gives an equivalence relation.

Therefore: The set of orbits of the action of G on S is a partition of S.

eg.  orbits are N, S poles; lines of latitude.

eg. what about orbits of $\mathbb{Z}/3\mathbb{Z} \curvearrowright$ globe?

<u>defn</u>. The <u>stabilizer</u> of s is the set of group elements that fix s:

$$G_s = \{g \in G \mid gs = s\} \subseteq G$$

<u>Claim</u> G_s is a subgroup of G. Why? Check id, inverse, closure.

eg.

1. In $S^1 \curvearrowright$ globe, G_N = all of S^1. But $G_p = \{1\}$ for any $p \neq S, N$.
2. $\text{Isom}(\mathbb{R}^2) \curvearrowright \mathbb{R}^2$: $G_{\vec{0}} \cong O(2)$
3. Stabilizer of n in $S_n \curvearrowright \{1, \dots, n\}$ is $\cong S_{n-1}$.

Properties of Group Actions $G \curvearrowright S$ based on orbits / stabilizers:

defns.

① If $G \curvearrowright S$ with just a single orbit, that means $\forall s, s' \in S$, there is a group element relating them: $s' = gs$
Then the action is called transitive

② The action of $G \curvearrowright S$ is free if
$\forall s \in S, \quad gs = s$ iff $g = 1$
ie only identity fixes elements.
In other words, $G \curvearrowright S$ is free if $\forall s \in S, G_s = \{1\}$.

Examples

① $\mathbb{Z}/3\mathbb{Z} \curvearrowright$ globe: not free (all g fix N, S)
not transitive (Interval worth of orbits)

② $Isom(\mathbb{R}^2) \curvearrowright$ not free: eg. $\tau(0) = 0$, but $\tau \neq 1$.
transitive: eg. $t_{b-a}(a) = b$.

③ H = subgroup of $T \leq Isom(\mathbb{R}^2)$ of horizontal translations.
free: $t_v(p) = p$ iff $v = \vec{0}$
not transitive: the orbits are the lines of constant y coordinate

④ $G \times G \longrightarrow G$ free and transitive

Participation Slip: Prove this

Remark: Very important to be clear about what set you are acting on.

eg. $S^1 \curvearrowleft$ points on the globe, or the lines of latitude

eg. If S is the set of Δs in the plane,

- stab. of a particular equilateral Δ is $\cong D_3$
- but the points of this Δ aren't fixed!

  (otherwise the only element in the stabilizer is 1...)

Claim Obvious that action of G is transitive on each orbit. Why?

Prop. 6.7.7 Let $G \curvearrowright S$, $s \in S$, G_s = stabilizer of s.

(a) If $a, b \in G$, then $as = bs$ iff $a^{-1}b \in G_s$, iff $b \in aG_s$

(b) Suppose $s' = as$. Then $G_{s'}$ is a conjugate subgroup to G_s

$$G_{s'} = aG_s a^{-1} = \{g \in G \mid g = aha^{-1} \text{ for some } h \in G_s\}.$$

Pf.

(a) is clear: $as = bs$ iff $a^{-1}bs = s$.

(b) $G_{s'} \supseteq aG_s a^{-1}$:

If $g \in aG_s a^{-1}$, then $g = aha^{-1}$ for some $h \in G_s$.

Then $gs' = (aha^{-1})(as) = ahs = as = s'$.

$G_{s'} \subseteq aG_s a^{-1}$:

Since $s = a^{-1}s'$, $a^{-1}G_{s'}a \subseteq G_s$ by the same argument.

$\Rightarrow G_{s'} \subseteq aG_s a^{-1}$.

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