Lecture 24

<u>Recall</u> An operation/action of a group G on a set S is a map

$G \times S \longrightarrow S$ where ① $1 \in G$ acts as identity map

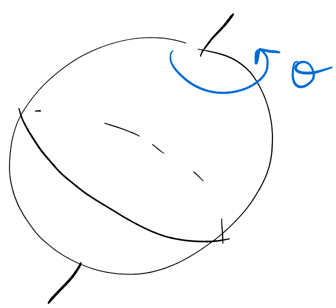
② $g'(g(s)) = (gg')(s)$ (associative)

Fix $s \in S$.

- The <u>orbit</u> of s is $O_s = \{s' \in S \mid s' = gs \text{ for some } g \in G\} = G * s$
- The <u>stabilizer</u> of s is $G_s = \{g \in G \mid gs = s\} \leq G$.

The action $G \curvearrowright S$ is
- <u>transitive</u> if there is only one orbit
- <u>free</u> if $(gs = s \Rightarrow g = 1)$

eg. $S^1 \curvearrowright$ globe



<u>Remark:</u> Very important to be clear about what set you are acting on.

eg. $S^1 \curvearrowright$ points on the globe, or the lines of latitude

eg. If S is the set of Δs in the plane,
- stab. of a particular equilateral Δ is $\cong D_3$
- but the points of this Δ aren't fixed!

(otherwise the only element in the stabilizer is 1...)

<u>Claim</u> Obvious that action of G is transitive on each orbit. Why?

Prop. 6.7.7 Let $G \curvearrowright S$, $s \in S$, G_s = stabilizer of s.

(a) If $a, b \in G$, then $as = bs$ iff $a^{-1}b \in G_s$, iff $b \in aG_s$

(b) Suppose $s' = as$. Then $G_{s'}$ is a conjugate subgroup to G_s

$$G_{s'} = aG_s a^{-1} = \{g \in G \mid g = aha^{-1} \text{ for some } h \in G_s\}.$$

Pf.

(a) is clear: $as = bs$ iff $a^{-1}bs = s$.

(b) $G_{s'} \supseteq aG_s a^{-1}$:

If $g \in aG_s a^{-1}$, then $g = aha^{-1}$ for some $h \in G_s$.

Then $gs' = (aha^{-1})(as) = ahs = as = s'$.

$G_{s'} \subseteq aG_s a^{-1}$:

Since $s = a^{-1}s'$, $a^{-1}G_{s'}a \subseteq G_s$ by the same argument.

$\Rightarrow G_{s'} \subseteq aG_s a^{-1}$.

$\square$

§6.8 Operation on Cosets (particular example of group action)

Recall $G \curvearrowright G$ by $G \times G \xrightarrow{\cdot} G$ (left multiplication)

Similarly, for $H \leq G$, $G \curvearrowright G/H$ = the set of left cosets of H

just a set, not a group unless $H \trianglelefteq G$!

$$G \times G/H \longrightarrow G/H$$
$$(g, [aH]) \longmapsto [gaH]$$

Write $[C]$ for the coset as an element of G/H.

prop $H \leq G$.

① $G \curvearrowright G/H$ is transitive

② $G_{[H]} = H$

$\boxed{\text{Participation Slip: give a short proof of this proposition (both parts).}}$

eg. $S_3 = \langle x, y \mid x^3 = y^2 = yxyx = 1 \rangle$ where $x = (1\ 2\ 3)$ $y = (2\ 3)$

Define $H = \{1, y\}$. Cosets: $C_1 = H = \{1, y\}$

$C_2 = xH = \{x, xy\}$

$C_3 = x^2H = \{x^2, x^2y\}$

The action of x and y, φ_x and φ_y, act on the indices of the cosets as $\varphi_x \leftrightarrow (1\,2\,3)$, $\varphi_y \leftrightarrow (2\,3)$.

Orbit-Stabilizer Theorem

$G \curvearrowright S$ can be described in terms of operations on cosets.

Prop. 6.8.4 (Orbit-Stabilizer theorem) Let $G \curvearrowright S$, $s \in S$.

There is a bijective map (of sets!)

$$\varepsilon : G/G_s \longrightarrow O_s \qquad [aH] \rightsquigarrow as$$

This map is compatible with the group action/operation, i.e.

$$\varepsilon(g[C]) = g\,\varepsilon([C]) \text{ for every coset } C \text{ and element } g \in G.$$

In other words: there is a G-equivariant set map

$$\varepsilon : G/G_s \longrightarrow O_s$$

$$\begin{tikzcd} G/G_s \arrow[r] \arrow[d, "g"'] \arrow[dr, phantom, "\circlearrowright"] & O_s \arrow[d, "g"] \\ G/G_s \arrow[r] & O_s \end{tikzcd}$$

} action of $g \in G$

Q. Why does this make sense?

|| $g \curvearrowright s \rightsquigarrow gs$, and all $g' \sim g$ do the same by defn.

Examples

① $D_5 \curvearrowright$ vertices of a regular pentagon. V.

Let $v \in V$. $H =$ stabilizer of v. Then there is a bijective map

$$\varepsilon : D_5/H \longrightarrow V$$

since $V =$ orbit of v; the action is transitive!

② $\mathrm{Isom}(\mathbb{R}^2) \curvearrowright \mathbb{R}^2$. $G_{\vec{0}} = O_2$. $\Rightarrow$ there is a bijection

$$\mathrm{Isom}(\mathbb{R}^2)/O_2 \longrightarrow P = O_{\vec{0}}. \text{ (right?)}$$

③ Let $\mathcal{L}$ denote the set of all lines in $\mathbb{R}^2$.

For $L \in \mathcal{L}$, let H_L denote the stabilizer of L.

Then $\mathrm{Isom}(\mathbb{R}^2)/H_L \longleftrightarrow \mathcal{L}$.

(Return to statement – why does the statement make sense? ie why would you believe this statement is true w/o seeing a full step-by-step proof?)

thm $G \curvearrowright S$, $s \in S$. $\quad \varepsilon : G/G_s \longleftrightarrow O_s$.

$[aH] \longleftrightarrow as$

Pf.

① ε is well-defined: Let $H = G_s$.

If $a, b \in G$ and $aH = bH$, then we must show $as = bs$.

If $aH = bH$, then $a^{-1}b \in H$, so $a^{-1}bs = s$. Then $bs = as$

(by assoc: $a \cdot a^{-1}bs = a \cdot s$!)

② ε^{-1} exists & is well-defined, and so ε is a bijection:

$\varepsilon^{-1}(as) = aH$. If $as = bs$, then $a^{-1}bs = s \Rightarrow a^{-1}b \in H \Rightarrow aH = bH$.

(same as above, because we are really using iff statements)

③ ε is G-equivariant.

$g \cdot \varepsilon(aH) = g \cdot as = (ga)s = \varepsilon(gaH)$. ✓ $\blacksquare$