# Lecture 25

Hand back exams @ end

## Orbit-Stabilizer Theorem

$G \curvearrowright S$ can be described in terms of operations on cosets.

Prop. 6.8.4 (Orbit-Stabilizer theorem) Let $G \curvearrowright S$, $s \in S$.

There is a bijective map (of sets!)

$$\varepsilon : G/G_s \longrightarrow O_s \qquad [aH] \rightsquigarrow as$$

This map is compatible with the group action/operation, i.e.

$$\varepsilon(g[C]) = g\,\varepsilon([C]) \text{ for every coset } C \text{ and element } g \in G.$$

In other words: there is a G-equivariant set map

$$\varepsilon : G/G_s \longrightarrow O_s$$

$$\begin{tikzcd}
G/G_s \arrow[r] \arrow[d, "g"'] \arrow[dr, phantom, "\circlearrowright"] & O_s \arrow[d, "g"] \\
G/G_s \arrow[r] & O_s
\end{tikzcd} \quad \Big\} \text{ action of } g \in G$$

Q. Why does this make sense?

|| $g \curvearrowright s \rightsquigarrow gs$, and all $g' \sim g$ do the same by defn.

### Examples

① $D_5 \curvearrowright$ vertices of a regular pentagon. $V$.

Let $v \in V$. $H$ = stabilizer of $v$. Then there is a bijective map

$$\varepsilon : D_5/H \longrightarrow V$$

since $V$ = orbit of $v$; the action is transitive!

② $\mathrm{Isom}(\mathbb{R}^2) \curvearrowright \mathbb{R}^2$. $G_{\vec{0}} = O_2$. $\Rightarrow$ there is a bijection

$$\mathrm{Isom}(\mathbb{R}^2)/O_2 \longrightarrow P = O_{\vec{0}}. \text{ (right?)}$$

③ Let $\mathcal{L}$ denote the set of all lines in $\mathbb{R}^2$.

For $L \in \mathcal{L}$, let $H_L$ denote the stabilizer of $L$.

Then $\mathrm{Isom}(\mathbb{R}^2)/H_L \longleftrightarrow \mathcal{L}$.

(Return to statement - why does the statement make sense? ie why would you believe this statement is true w/o seeing a full step-by-step proof?)

<u>thm</u> $G \curvearrowright S$, $s \in S$. $\varepsilon: G/G_s \longleftrightarrow O_s$.

$[aH] \longleftrightarrow as$

Pf.

① $\varepsilon$ is well-defined: Let $H = G_s$.

If $a, b \in G$ and $aH = bH$, then we must show $as = bs$.

If $aH = bH$, then $a^{-1}b \in H$, so $a^{-1}bs = s$. Then $bs = as$

(by assoc: $a \cdot a^{-1}bs = a \cdot s$!)

② $\varepsilon^{-1}$ exists & is well-defined, and so $\varepsilon$ is a bijection:

$\varepsilon^{-1}(as) = aH$. If $as = bs$, then $a^{-1}bs = s \Rightarrow a^{-1}b \in H \Rightarrow aH = bH$.

(same as above, because we are really using iff statements)

③ $\varepsilon$ is $G$-equivariant.

$g \cdot \varepsilon(aH) = g \cdot as = (ga)s = \varepsilon(gaH)$. ✓ ▨

# The Counting Formula: Consequence of Orbit-Stabilizer

Now consider $G$ = finite group, $H \leq G$.

Recall ① $[G:H] = |G/H|$ ← set of left cosets of $H$

② # elements in each coset is the same.
(Do you remember / know why?)

Counting formula: $|G| = |H| \underbrace{|G/H|}_{[G:H]}$

Similarly:

Prop 6.9.2 Let $S$ be a finite set, on which a group $G$ acts.

$G \curvearrowright S$ ← finite

For fixed $s \in S$: $|G| = |G_s| |O_s|$

Pf. $|G| = |G_s| \underbrace{|G/G_s|}$

but $G/G_s \longleftrightarrow O_s$ by orbit-stab ∎

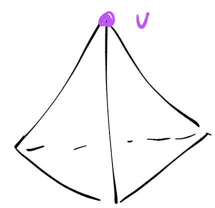
Rmks • ie $[G:G_s] = O_s$

• $|O_s| \mid |G|$

## Using Counting Formula to understand how orbits partition finite $S$:

$$|S| = |O_1| + |O_2| + \cdots + |O_k| \quad \text{where each } |O_i| \mid |G| \,!$$

eg. Let $G$ be the set of rotational symmetries of a tetrahedron

$T =$ 

in $SO(3)$: orientation-preserving rigid motions / isometries

$T$ has vertices $V$, edges $E$, faces $F$

$|V| = 4$, $|E| = 6$, $|F| = 4$.

We can fix a vertex $v$ and consider the subgroup $G_v$

We can restrict the action $G \curvearrowright T$ to an action $G_v \curvearrowright T$

$\cong \mathbb{Z}/3\mathbb{Z}$ ! (later)

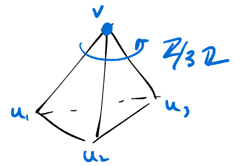
(because we can always restrict to the action of a subgroup; think $\mathbb{Z}/3\mathbb{Z} \leq S^1$ on globe)

Furthermore, we can study how this action induces an action on each of the sets $V, E, F$.

Tetrahedra (Regular)

define tetrahedra?

$G_v \circlearrowright V$ : $|V| = |\{v\}| + |\{u_1, u_2, u_3\}| = \underbrace{1 + 3}$.




each divides $|G| = |\mathbb{Z}/3\mathbb{Z}| = 3$.

Participation: Write down formulas like ⊛

for (a) $G_v \circlearrowright E$ (b) $G_v \circlearrowright F$.

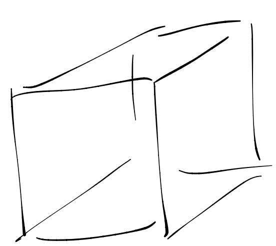
(You can name the edges/faces, or describe carefully which ones you are talking about.)

Also: $G_e \cong$ ? These also each act on

$G_f \cong$ ? V, E, F.

⇒ w/ restrictions and inductions, we have 9 group actions we can study.

On HW08, you'll study all these for the cube:



## §6.10 Operation on subsets

Stabilizer of a subset $U$ is

set of elements where $[gU] = [U]$
ie $gU = U$
(ie $\forall\, u \in U,\ gu \in U$)

eg. let $O$ denote the octahedral group:

24 rotations of the cube

Let $F$ = faces (6 of them)

$O$ then also acts on the sets of unordered pairs of faces $\binom{6}{2} = \frac{6!}{2!4!} = \frac{6 \cdot 5}{2} = 15$

$15 = \underbrace{\{\text{pairs of opp. faces}\}}_{3} \cup \underbrace{\{\text{pairs of adjacent faces}\}}_{12}$

Hand back exams.