Lecture 26

There are 4 more lectures : Chp. 7 material

Rough Plan
- (F) Cayley, Conj. action
- (M) p-groups
- (W) Sylow
- (F) Final Review

No HW assigned for this material, but there will be suggested exercises you should think through and some solutions will be provided.
These sections ≈ review of the course

§7.1 Cayley's Theorem

<u>Recall</u> $G \curvearrowright G$ by left multiplication
$$G \times G \longrightarrow G$$
$$(g, x) \longmapsto gx$$

This action is transitive + free

both: the action is <u>faithful</u> define

The permutation <u>representation</u>
$$G \longrightarrow \mathrm{Perm}(G)$$
$$g \longmapsto m_g \quad = \text{left mult by } g$$
defined by this action is <u>injective</u>

<u>aside</u> A <u>representation</u> of a group usually means a homomorphism $G \longrightarrow \underbrace{\mathrm{End}(V)}_{\text{a group!}}$ where V is a vector space.

* eg. Using matrices $(= \text{linear maps } \mathbb{R}^n \to \mathbb{R}^n = \mathrm{End}_{\mathbb{R}}(\mathbb{R}^n))$ to represent the Klein 4 group

Here, the terminology is akin: $G \longrightarrow \underbrace{\mathrm{End}_{Set}(G)}_{\text{a group}} \cong \mathrm{Perm}(G)$

thm (Cayley's Theorem)

Every finite group G is isomorphic to a subgroup of some permutation group S_n.

Pf. Let $n = |G|$. Then $\mathrm{Perm}(G) \cong S_n$. The homomorphism

$$\begin{aligned} \varphi : G &\longrightarrow \mathrm{Perm}(G) \cong S_n \\ g &\longmapsto m_g \end{aligned}$$

is injective, so $G \cong \mathrm{Im}\,\varphi$. ▨

One interesting question (we will not answer)

If $G \hookrightarrow S_n$, then $G \hookrightarrow S_m$ $\forall$ $m \geq n$.

What is the smallest n such that a given group G injects into S_n?

There are many theorems of this kind in math:

eg. all finite-dim'l manifolds embed into some $\mathbb{R}^n$

But Klein bottle (a 2D manifold) embeds only into $\mathbb{R}^4$ and higher!

§7.2 The Class Equation

$G \curvearrowright G$ by conjugation as well: $\quad \begin{array}{c} G \times G \longrightarrow G \\ (g, x) \longmapsto gxg^{-1} \end{array}$

In this section we write $g * x$ to mean this action: $\quad g * x = gxg^{-1}$.

You've already seen in this class how important the action of conjugation is.
We have special terms for stabilizers & orbits for this action.

Consider $G \curvearrowright G$ by conjugation. Let $x \in G$.

<u>defn</u>. The centralizer of x, $Z(x)$ = stabilizer of x.
from German Zentral

$= \{ g \in G \mid gxg^{-1} = x \}$

$= \{ g$ that commute with $x \}$ Check this! Does this make sense to you?

In general, the word "central" means something about commutation:

<u>Recall</u> The Center $Z(G)$ of a group is
$\{g$ that commute w/ all other $x \in G\} = \bigcap_{x \in G} Z(x)$

eg. $Z(G) = G$ for abelian G.
$Z(S_n) = 1$.

<u>Review</u> Prove that $Z(G)$ is a subgroup of G.

(defn) The <u>conjugacy class</u> of $x \in G$ is the orbit of x.

eg. We first saw "conj class" in the context of S_n, when we saw that

$$\left\{ \begin{matrix} \text{cycle} \\ \text{types} \end{matrix} \right\} \longleftrightarrow \left\{ \begin{matrix} \text{conj} \\ \text{classes} \end{matrix} \right\}$$

The Counting Formula then tells us:

$$|G| = |G_x|\,[G:G_x] = |G_x|\,|O_x| = |Z(x)|\,\underbrace{|\text{conj class}(x)|}_{\substack{\text{call this} \\ C(x)}}$$

Immediate Observations:

① $x \in Z(x)$ x commutes w/ itself

infact, $\langle x \rangle \subset Z(x)$. for many reasons:

- $Z(x)$ is a group
- x^n commutes w/ x

② $Z(G) \leq Z(x)$

③ an element $x \in G$ is in $Z(G)$ (ie comm w/ all)

iff $Z(x) = G$

iff its conj class $C(x) = \{x\}$

Think through this.

The Class Equation

⇝ how conjugacy classes partition G

<u>defn.</u> For a finite group G, the <u>class equation of G</u> is

$$|G| = \sum_{\substack{\text{conj}\\ \text{classes } C}} |C| = |C_1| + |C_2| + \cdots + |C_k|.$$

↑ since $1 \in Z(G)$, $C(1) = \{1\}$ & we default to letting $C_1 = C(1)$.

* B/c this is decomp of G into orbits, counting formula ⇒ all the $|C_i| \mid |G|$.

eg. Class equation of $S_3 \cong D_3$:

$$6 = 1 + 2 + 3$$

$$S_3 = \{1\} \cup \underbrace{\{(1\,2\,3), (1\,3\,2)\}}_{= A_n,\ \text{a normal subgroup!}} \cup \{(1\,2), (2\,3), (1\,3)\}$$