Lecture 27

<u>Recall</u> The class equation of a finite group G is

$$|G| = |C_1| + |C_2| + \cdots + |C_k|$$

conjugacy classes of G

eg. $D_4 = \langle \rho, \tau \mid \rho^4 = \tau^2 = \rho\tau\rho\tau = 1 \rangle = G$

- $|Z(\rho)| = 4$

  <u>Why?</u> <u>Participation</u>

  $|D_4| = 2n \Rightarrow |Z(\rho)|$ divides $2n$.

  $\langle \rho \rangle \subset Z(\rho) \Rightarrow |Z(\rho)| \geq n$.

  But $\tau \notin Z(\rho) \Rightarrow |Z(\rho)| = n$.

$\Rightarrow |C(\rho)| = 2$. (The other conjugate is ρ^3.)

- What about ρ^2? Clearly $\rho^k \rho^2 \rho^{-k} = \rho^2$.

  Compute $\tau\rho^2\tau = \tau\rho\rho\tau = \rho^{-1}\tau\rho\tau = \rho^{-1}\rho^{-1}\tau\tau = \rho^{-2} = \rho^2$.

  $\Rightarrow Z(\rho^2) = D_4 \Rightarrow |C(\rho^2)| = 1$. $\quad$ ($\rho^2 \in Z(G)$!)

- At this point, we have

$$|D_4| = 8 = \underset{C(1)}{1} + \underset{C(\rho^2)}{1} + \underset{C(\rho)}{2} + \underline{\ \ ?\ \ }$$

- Consider $\rho\tau$: What is $(\rho^k\tau)^{-1}$? $= \rho^k\tau$!

  Compute centralizer:

  For $0 \le k < 4$.

  - $\rho^k(\rho\tau)\rho^{-k} = \rho^{k+1}\rho^k\tau = \rho\tau$ iff $\rho^{2k+1} = \rho$

    ie $2k+1 \equiv 1 \pmod 4 \iff 2k \equiv 0 \pmod 4$

    $\Rightarrow k \in \{0, 2\}$

  - $(\rho^k\tau)(\rho\tau)(\rho^k\tau) = \rho^{k-1}\cdot\rho^k\tau = \rho^{2k-1}\tau = \rho\tau$

    iff $\rho^{2k-1} = \rho$ ie $2k-1 \equiv 1 \bmod 4$ ie $2k \equiv 2 \Rightarrow k \in \{1, 3\}$

  So $|Z(\rho\tau)| = 4 \Rightarrow |C(\rho\tau)| = 2$.

- So $8 = 1 + 1 + 2 + 2 + 2$... we're done!

Q. So how is the class equation used?

(A lot to understand representation... but here are examples we are already equipped to understand)

§7.3 p-Groups

defn A p-group is a finite group of order p^r for some $r \in \mathbb{N}$.

prop 7.3.1 The center $Z(G)$ of a p-group G is not the trivial group.

Pf.

Participation Slip ?: How would you prove this?
Hint: Use the class equation.

Suppose $|G| = p^r$ with $r \geq 1$.

$\Rightarrow$ every $|C| \in \{1, p, p^2, \ldots, p^r\}$.

We know $|C_1| = |C(1)| = 1$. Then

$p^r = 1 + \sum(\text{multiples of } p)$.

$\Rightarrow$ There must be more 1's on the right.

$\Rightarrow Z(G) \neq \{1\}$. $\blacksquare$

Similar story for group actions:

defn. Let $G \curvearrowright S$. If $G_s = G$, then $s \in S$ is a fixed point of the action.

eg. $S^1 \curvearrowright$ globe poles = fixed points

thm (Fixed point theorem)

Let G be a p-group. Let S be a finite set on which G acts.

If $p \nmid |S|$, then there is a fixed point of the action $G \curvearrowright S$.

Pf. Review problem!

__prop.__ Every group of order p^2 is abelian

pf. Let G be a group of order p^2.

By prev. prop, $Z(G) \neq \{1\} \Rightarrow |Z(G)| = p$ or p^2.

If $|Z(G)| = p^2$, then $Z(G) = G \Rightarrow G$ abelian.

So it remains to show $|Z(x)| \neq p$.

BWOC, suppose $|Z(x)| = p$. Then let $x \notin Z(G)$.

Both $\langle x \rangle \leq Z(x)$ and $Z(G) \leq Z(x)$, so

$\{x\} \cup Z(G) \subset Z(x)$, so $Z(x) \supsetneq Z(G)$

$\Rightarrow Z(x) = G$ (it must have order p^2)

But then $x \in Z(G)$ as it commutes with all of G... $\lightning$. $\blacksquare$

Cor. A group of order p^2 is either cyclic ($\cong C_{p^2}$).
or the product of two cyclic groups of order p ($\cong C_p \times C_p$).

Pf. (Not actually very immediate because we did a lot of work understanding product grps in chp 2!)

Let G be of order p^2.

- If G has an order p^2 element x, then $G = \langle x \rangle \cong C_{p^2}$.
- If not, then there is an element of order p (only 1 has order 1); call this x.

  Pick $y \in G \setminus \langle x \rangle$; it must also have order p.

Then ① $\langle x \rangle \cap \langle y \rangle = 1$

BWOC suppose $1 \neq y^n = x^m$.

Since $|\langle y \rangle| = p$ is prime, $|\langle y^n \rangle| = p$ so $\langle y^n \rangle = \langle y \rangle$.

So there is some $k \in \mathbb{N}$ s.t. $(y^n)^k = y \Rightarrow y = x^{mk}$.

But $y \notin \langle x \rangle$. $\lightning$.

② $\langle x \rangle, \langle y \rangle \trianglelefteq G$ b/c G abelian.

③ $HK = G$

Since $K, H \subseteq HK$, we know $\langle x \rangle \cup \langle y \rangle \subset HK$.

By this is already $2p-1$ elements, so HK must contain the whole group.

Recall: $H \trianglelefteq G \Rightarrow HK$ is a subgroup + must have order $1, p,$ or p^2. $\boxed{Z}$