Lecture 28

Announcements:

1. Final Study Guide online
   - will talk more about this on Friday (review day)
2. Teaching evals (date due?)

Recall from last time:

Prop. Every group of order p^2 is abelian

Cor. A group of order p^2 is either cyclic ($\cong C_{p^2}$).
or the product of two cyclic groups of order p ($\cong C_p \times C_p$).

Pf. (Not actually very immediate because we did a lot of work understanding product grps in chp 2!)

Let G be of order p^2.

- If G has an order p^2 element x, then $G = \langle x \rangle \cong C_{p^2}$.
- If not, then there is an element of order p (only 1 has order 1); call this x.

  Pick $y \in G \setminus \langle x \rangle$; it must also have order p.

Then ① $\langle x \rangle \cap \langle y \rangle = 1$

Prop 2.11.4: recognizing products.

BWOC suppose $1 \neq y^n = x^m$.

Since $|\langle y \rangle| = p$ is prime, $|\langle y^n \rangle| = p$ so $\langle y^n \rangle = \langle y \rangle$.

So there is some $k \in \mathbb{N}$ s.t. $(y^n)^k = y \Rightarrow y = x^{mk}$.

But $y \notin \langle x \rangle$. ↯.

② $\langle x \rangle, \langle y \rangle \trianglelefteq G$ b/c G abelian.

③ $HK = G$

Since $K, H \subseteq HK$, we know $\langle x \rangle \cup \langle y \rangle \subset HK$.

By this is already $2p-1$ elements, so HK must contain the whole group.

Recall: $H \trianglelefteq G \Rightarrow HK$ is a subgroup + must have order $1, p,$ or p^2.

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§7.7 The Sylow Theorems

↑ SEE-luv but we've been saying see-low for too long...

Throughout: $|G| = n$.

Idea: Study an arbitrary finite group ($|G| = n$) by studying subgroups that are of order p^r, where r is the largest power of p that divides n.

$n = p^r m \qquad p \nmid m.$

If $H \leq G$ has $|H| = p^r$, then H is a Sylow p-subgroup of G

In other words: A Sylow p-subgroup is a p-group whose index is not divisible by p. ("maximal p-subgroup")

Now use counting formula:

$$\underbrace{p^r}_{=0 \bmod p}\underbrace{m}_{\neq 0 \bmod p} = |G| = \underbrace{|G_{[u]}|}_{\overset{?}{=}0 \bmod p} \cdot \underbrace{|O_{[u]}|}_{\neq 0 \bmod p}$$

$\Rightarrow |O_{[u]}| = m$

$\Rightarrow |G_{[u]}| = p^r$. We found it!

<u>Cor. to Sylow I.</u> A finite group whose order is divisible by a prime p contain an element of order p.

Let's just think through why... $|H| = p^r \rightsquigarrow$ find elt. //

eg. If $|G| = 6$, the elements can't all have order 1 or 2.

There are 2 more Sylow theorems, and their proofs are similar in length + technique: considering group actions. These use the conjugation action though.

(Good practice to see how well you understand conj action on group to work through the proofs.)

2nd ~~Sylow Theorem~~ Sylow II Let G be finite group w/ $p \mid n = |G|$.

(a) all Sylow p-subgroups are conjugate subgroups.

ie the conjugation action of G on the set $\{$ Sylow p-subgroups of $G\}$ is transitive.

(b) Every subgroup of G that is a p-group is contained in a Sylow p-subgroup.

(Note if $|H| = p^r \Rightarrow |gHg^{-1}| = p^r$ as well.)

Cor. G has exactly one Sylow p-subgroup H $\Leftrightarrow$ that H is normal in G. (why?)

3rd ~~Sylow theorem~~ Sylow III (with same setup as throughout:)

$|G| = n = p^r m$, $p \nmid m$. Let $s = \#$ Sylow p-subgroups

Then $s \mid m$, and $s \equiv 1 \pmod{p}$.

Q Why would this be useful?

Example / Prop. Every group of order 15 is cyclic.

($\Rightarrow$ there's only one isom class!)

Pf.

Let $|G| = 15 = 3 \times 5$.

Let $s_3 = \#$ Sylow 3-subgroups.

$\text{III} \Rightarrow \quad s_3 \mid 5, \; s_3 \equiv 1 \pmod 3) \quad \Rightarrow s_3 = 1.$

$\Rightarrow$ the unique sylow 3 subgroup H is normal.

Let $s_5 = \#$ Sylow 5 subgroups

$\text{III} \Rightarrow \quad s_5 \mid 3, \; s_5 \equiv 1 \pmod 5)$

$\Rightarrow \exists! \; K \leq G, \; |K| = 5, \; K \trianglelefteq G.$

Since $|H| = 3$ & $|K| = 5$, $H \cap K = \{1\}$.

(& HK must have order $15 \Rightarrow HK = G$).

Prop 2.11.4 $\Rightarrow G \cong H \times K \cong C_3 \times C_5 \cong C_{15}$. ∎