LECTURE 3

- HW01 is up
- Xuxing covering next Wed & Fri.

Q. Pseudocode where you access the matrix columnwise?

Last time Matrix-vector multiplication ① Symbolically ② Pseudocode

③ Meaning: a "linear" transformation

"Linearity" A function $f: \mathbb{R}^n \longrightarrow \mathbb{R}^m$ is ($\mathbb{R}$-) linear if

$\forall\, c \in \mathbb{R},\ x \in \mathbb{R}^n, \qquad f(c \cdot x) = c \cdot f(x)$

Intuition / visually: Shear, stretch, reflect
no translation or twisting.

Matrix-matrix multiplication

① Symbolically: $A \in \mathbb{R}^{m \times k},\ B \in \mathbb{R}^{k \times n} \Rightarrow AB \in \mathbb{R}^{m \times n}$

$$m\left\{\underbrace{\begin{bmatrix} \ \ A\ \ \end{bmatrix}}_{k}\right. \left.\underbrace{\begin{bmatrix} \ \ \ B\ \ \ \end{bmatrix}}_{n}\right\}k = m\underset{n}{\begin{bmatrix} \ \ C = AB\ \ \end{bmatrix}}$$

$$c_{ij} = \sum_{s=1}^{k} a_{is} \cdot b_{sj} \qquad i \in [m],\ j \in [n]$$

② Pseudocode

③ Example & meanings

- Action on column / row space:

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 3 & 4 \\ 1 & 2 \end{bmatrix} \qquad \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 4 & 3 \end{bmatrix}$$

- As linear transformations: composition of functions.

$$A \in \mathbb{R}^{m \times k},\ B \in \mathbb{R}^{k \times n} \implies \mathbb{R}^n \xrightarrow{B} \mathbb{R}^k \xrightarrow{A} \mathbb{R}^m$$

So $(AB)v = A\big(B(v)\big)$.

- <u>Outer product</u> of vectors $x \in \mathbb{R}^m$, $y \in \mathbb{R}^n$

$$xy^T = \begin{bmatrix} x_1 \\ \vdots \\ x_m \end{bmatrix}\begin{bmatrix} y_1 & \cdots & y_n \end{bmatrix} = \begin{bmatrix} x_1y_1 & \cdots & x_1y_n \\ \vdots & \ddots & \vdots \\ x_1y_n & \cdots & x_my_n \end{bmatrix}$$

- Suppose $C = AB$

Outer product form of matrix multiplication

$$C = AB = \begin{bmatrix} a_{\cdot 1} & a_{\cdot 2} & \cdots & a_{\cdot k} \end{bmatrix}\begin{bmatrix} b_{1\cdot}^T \\ b_{2\cdot}^T \\ \vdots \\ b_{k\cdot}^T \end{bmatrix} = \underbrace{\sum_{s=1}^{k} a_{\cdot s} b_{s\cdot}^T}_{\text{Expansion of } C \text{ by simple matrices}}$$

<u>Simple matrices</u>: $a_{\cdot s} b_{s\cdot}^T$ are outer products of matrices.

(These are rank 1 matrices... why? Review rank!)

- Inner product: $x^T y \in \mathbb{R}$. "dot product" - measure of similarity

eg. $\begin{bmatrix}3 & 1 & 2\end{bmatrix}\begin{bmatrix}0\\1\\-1\end{bmatrix} = 0+1-2 = -1.$

* $\|v\|_2 = \sqrt{v \cdot v}$ measure of size of vector by Euclidean norm.

"Similarity" between two vectors: cosine of the angle between two vectors
a "distance" measure

$$\cos\theta(x,y) = \frac{x^T y}{\|x\|_2 \|y\|_2}$$

* HW01 will involve this

eg. Term-document matrix, query vector

$$\begin{array}{c|c} & D_1\ D_2\ D_3\ D_4\ D_5\ D_6 \\ \hline \begin{matrix}T_1\\T_2\\T_3\\T_4\end{matrix} & \begin{bmatrix} & & \\ & A & \\ & & \end{bmatrix}\end{array} \qquad \begin{array}{c|c} & q_1\ q_2\ q_3 \\ \hline \begin{matrix}T_1\\T_2\\T_3\\T_4\end{matrix} & \begin{bmatrix} & & \\ & Q & \\ & & \end{bmatrix}\end{array}$$

$$A^T q_1 = \begin{array}{cc} T_1\ T_2\ T_3\ T_4 & \\ \begin{bmatrix} \ \ & & & \\ & & & \\ & & & \\ & & & \end{bmatrix} & \begin{matrix} D_1 \\ D_2 \\ \vdots \\ \ \end{matrix}\end{array} \begin{array}{c} q_1 \\ \begin{bmatrix} \ \\ \ \\ \ \end{bmatrix}\end{array} = \begin{bmatrix} \ \\ \ \\ \ \\ \ \end{bmatrix} \begin{matrix} \leftarrow D_1 \cdot q_1 \\ \leftarrow D_2 \cdot q_1 \\ \\ \\ \end{matrix}$$

similarity scores

* q_j is itself a document!

$A^T Q$: entry i,j represent similarity b/w D_i and q_j.

(Bonus) Notions of size for vectors, matrices

$$\|x\|_1 = \sum_{i=1}^{n} |x_i| \qquad \|x\|_p = \sqrt[p]{\sum_{i=1}^{n} x_i^p} \qquad \|x\|_\infty = \max_{i \in [n]} |x_i|$$

"1-norm" "L_p-norm" "max-norm"

computation-heavy:

$$\|A\|_2 = \Big(\underbrace{\max_{i \in [n]} \lambda_i (A^T A)}_{\text{largest eigenvalue of } A^T A}\Big)^{1/2}$$