LECTURE 11

Q. Prove that orthogonal transformations preserve length:

$Q \in \mathbb{R}^{n\times n}$ orthogonal, $x \in \mathbb{R}^n$

Show that $\|Qx\|_2 = \|x\|_2$. (square both sides)

Geometric properties of orth. matrix Q viewed as a linear trans

- length-preserving

- The corresponding operator (matrix) norm + Frob norm also preserved: in the following sense:

prop $U \in \mathbb{R}^{m\times m}$, $V \in \mathbb{R}^{n\times n}$ orthogonal. For any $A \in \mathbb{R}^{m\times n}$,

(a) $\|UAV\|_2 = \|A\|_2$

(b) $\|UAV\|_F = \|A\|_F$

Pf.

(a) Let $\|\cdot\| = \|\cdot\|_2$.

$$\|UAV\| = \sup_{x\neq 0} \frac{\|UAVx\|}{\|x\|} = \sup_{x\neq 0} \frac{\|AVx\|}{\|x\|} = \sup_{x\neq 0} \frac{\|A(Vx)\|}{\|Vx\|} \underset{V \text{ invertible}}{=} \sup_{y\neq 0} \frac{\|Ay\|}{\|y\|} = \|A\|.$$

(Why we needed $\|\cdot\|_2$ specifically in this proof?

(b) $\|A\|_F^2 = tr(A^TA)$ (recall)

$$\|UAV\|_F^2 = tr\left(\underbrace{(UAV)^T(UAV)}_{= V^TA^TU^TUAV \,=\, V^TA^TAV}\right) = tr(VV^TA^TA) = tr(A^TA).$$

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§4.2 Elementary Orthogonal Matrices

- Use the elementary pieces to reduce matrices to compact form.

eg. $A \in \mathbb{R}^{m\times n}$ $(m \geq n)$ $\rightsquigarrow$ triangular form.

Imagine length-preserving transformations of 2D or 3D space... What do these look like?

Ⓘ Plane Rotation (2D)

2×2 plane rotation matrix: "Givens rotations"

$$G = \begin{pmatrix} c & s \\ -s & c \end{pmatrix} \qquad c^2 + s^2 = 1$$

Q. Why did I choose the variables c and s?
Let's see what happens to a vector:

$\theta = 90^\circ = \frac{\pi}{2}$. $(\cos\theta, \sin\theta) = (0, 1)$.

$$G = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \end{pmatrix} \qquad \vec{e}_1 \rightsquigarrow -e_2$$

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad e_2 \rightsquigarrow \vec{e}_1$$

I told you these are rotations ⇒ $G(\theta)$ rotates vectors CW by $\frac{\pi}{2}$.

In a higher ambient space, we can still do a plane rotation:

eg. $G \in \mathbb{R}^{4\times 4}$ id on e_1, e_3 but rotation in plane $\langle e_2, e_4 \rangle$:

$$G = \begin{bmatrix} 1 & & & \\ & c & & s \\ & & 1 & \\ & -s & & c \end{bmatrix}$$

How useful?

We can transform any vector to a multiple of a unit vector of the form e_i.

(Agreed?)

Let's see how to carry this out explicitly:

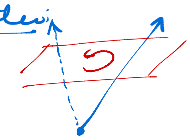
e.g. Given $x \in \mathbb{R}^4$, want to transform to κe_1.

$$x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} \overset{G_3}{\rightsquigarrow} \begin{pmatrix} x_1 \\ x_2 \\ x_3' \\ 0 \end{pmatrix}$$

id here

rotated the part in this plane.

schematic



$$\overset{G_2}{\rightsquigarrow} \begin{pmatrix} x_1 \\ x_2' \\ 0 \\ 0 \end{pmatrix} \overset{G_1}{\rightsquigarrow} \begin{pmatrix} x_1' \\ 0 \\ 0 \\ 0 \end{pmatrix} = \kappa e_1 \text{ for some } \kappa \in \mathbb{R}.$$

Since the G_i are orthogonal, $P = G_1 G_2 G_3$ is orthogonal

$\Rightarrow$ length preserving.

From another point of view, you're just turning your head & looking at the same vector:

$$\begin{tikzcd} \mathbb{R}^4 \arrow[r, bend left, "G_1 G_2 G_3", "\cong"'] & \mathbb{R}^4 \arrow[l, bend left, "(G_1 G_2 G_3)^{-1}"] \end{tikzcd}$$

II) Reflection matrices "Householder transformations"

Given 2 vectors of the same length, there is a "reflection" relating them:

e.g. In 2D: across a line

In 3D: across a plane

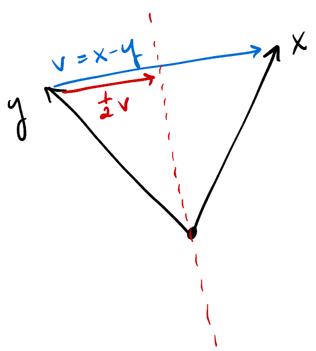
$A = \sum_i \lambda_i q_i q_i^T$ $\{q_i\}$ ON basis

$\Rightarrow$ given x, A is

$x = \sum \alpha_i q_i \mapsto \sum \lambda_i \alpha_i q_i$

Note Across a subspace 1-dim lower than ambient space. What is x, y's relation to this subspace?

A: Same distance



Want: $v \neq 0$

rank 1 matrix: $x = x' + \alpha v \mapsto x' - \alpha v$, $x' \perp v$.

$$P = I - \underbrace{\frac{2}{v^T v} v v^T}_{\text{square mat}}$$ } defn of Householder trans

s.t. $Px = y$?

With $v = x - y$, $v^T v = (x-y)^T(x-y) = (x^T - y^T)(x-y)$

$= x^T x - y^T x - x^T y + y^T y$

$= 2(x^T x - x^T y)$ $\|x\|_2 = \|y\|_2$; dot product

$\Rightarrow v^T x = (x^T - y^T)x = x^T x - y^T x = \frac{1}{2} v^T v$ from above

$\Rightarrow Px = x - \frac{2 v^T x}{v^T v} v = x - \frac{\cancel{2(\frac{1}{2}v^T v)}^{1}}{\cancel{v^T v}} v = y$ indeed!

Now simplify by normalizing: $u = \frac{v}{\|v\|_2}$ (unit vector)

$P = I - \frac{2}{v^T v} v v^T = I - 2uu^T$

This unit vector u is the Householder vector

& we can compute it in MATLAB (you'll do this on HW)