LECTURE 13

<u>Review of Projectors</u>   <u>Note</u> range = image, = colspace in our case

- A matrix $P \in \mathbb{R}^{m \times m}$ (i.e. $P: \mathbb{R}^m \to \mathbb{R}^m$) is a <u>projector</u> if $P^2 = P$.
- For all $x \in \mathbb{R}^m$, $x - Px \in \text{null}(P)$: $\quad P(x - Px) = Px - P^2x = Px - Px = 0$
- If P is a projector, then $I-P$ is also a projector:

$$(I-P)^2 = I^2 - IP - PI + P^2 = I - P - P + P = I - P.$$

$I-P$ is the <u>complementary projector to P.</u>

<u>thm</u> (a) $\text{range}(I-P) = \text{null}(P)$

(b) $\text{null}(I-P) = \text{range}(P)$

Pf.

(a)

- $\text{null}(P) \subset \text{range}(I-P)$:

  Suppose $v \in \text{null}(P)$. Then $(I-P)v = v - Pv = v \Rightarrow v \in \text{range}(I-P)$.

- $\text{range}(I-P) \subset \text{null}(P)$:

  Suppose $v \in \text{range}(I-P)$. Then $\exists\ x \in \mathbb{R}^m$ such that $(I-P)x = v$

  $\Rightarrow Pv = P(I-P)x = Px - Px = 0 \Rightarrow v \in \text{null}(P)$.

(b) This follows from (a): use $P' = I - P$. Then $I - P' = I - (I-P) = P$.

∎

<u>Remarks</u>

- $\text{range}(P) \cap \text{null}(P) = \{0\} \Rightarrow \quad \text{null}(P) \cap \text{null}(I-P) = \{0\}$

  $\text{range}(P) \cap \text{range}(I \cap P) = \{0\}$.

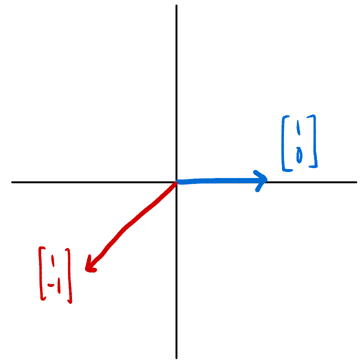
<u>Note</u>: $P + (I-P) = I \Rightarrow (Px) + ((I-P)x) = x$

$\rightsquigarrow$ P and $I-P$ really are complementary eg.

$$P = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \qquad I-P = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 0 & 1 \end{bmatrix}$$

- check that $P^2 = P$, $(I-P)^2 = (I-P)$.
- $\text{range}(P) = \text{colspace}(P) = \text{span}\left\{\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right\}$

  $\text{null}(P) = \text{range}(I-P) = \text{span}\left\{\begin{bmatrix} -1 \\ 1 \end{bmatrix}\right\}$



Note that $\mathbb{R}^2 = \text{Span}\left\{\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right\} + \text{span}\left\{\begin{bmatrix} -1 \\ 1 \end{bmatrix}\right\}$

but $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$ are not orthogonal.

<u>defn</u>. A projector P is <u>orthogonal</u> if range $P \perp \text{null}(P)$.

eg. $P = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$

⚠ P is <u>NOT</u> an orthogonal matrix! It's not even invertible.

<u>thm</u> A projector P is an orthogonal projector iff $P^T = P$ (ie symmetric).

<u>Pf</u> $\boxed{\Leftarrow}$ Let $v_1 \in \text{range}(P)$, $v_2 \in \text{null}(P)$.

Since $v_1 \in \text{range}(P)$, $\exists$ x such that $Px = v_1$.

Then $v_1 \cdot v_2 = (Px) \cdot v_2 = (Px)^T v_2 = x^T P^T v_2 = x^T P v_2 = x^T \vec{0} = 0$.

$\Rightarrow v_1 \perp v_2$ indeed.

$\boxed{\Rightarrow}$ Omitted. (Harder)

II) Reflection matrices "Householder transformations"

Given 2 vectors of the same length, there is a "reflection" relating them:

eg. In 2D: across a line

In 3D: across a plane

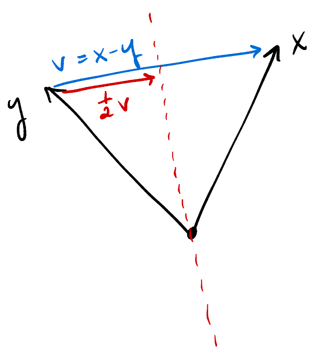
$A = \sum_i \lambda_i q_i q_i^T$ $\{q_i\}$ ON basis

$\Rightarrow$ given x, A is

$x = \sum \alpha_i q_i \mapsto \sum \lambda_i \alpha_i q_i$

Note Across a subspace 1-dim lower than ambient space. What is x, y's relation to this subspace?

A: Same distance



Want: $v \neq 0$

rank 1 matrix: $x = x' + \alpha v \mapsto x' - \alpha v$, $x' \perp v$

$$P = I - \frac{2}{v^T v} v v^T$$

square mat

} defn of Householder trans

s.t. $Px = y$?

With $v = x - y$, $\quad v^T v = (x-y)^T(x-y) = (x^T - y^T)(x-y)$

$$= x^T x - y^T x - x^T y + y^T y$$

$$= 2(x^T x - x^T y) \qquad \|x\|_2 = \|y\|_2\text{; dot product}$$

$\Rightarrow v^T x = (x^T - y^T)x = x^T x - y^T x = \frac{1}{2} v^T v$ from above

$\Rightarrow Px = x - \frac{2 v^T x}{v^T v} v = x - \frac{\cancel{2(\frac{1}{2} v^T v)}^{1}}{\cancel{v^T v}} v = y$ indeed!

Now simplify by normalizing: $u = \frac{v}{\|v\|_2}$ (unit vector)

$$P = I - \frac{2}{v^T v} v v^T = I - 2uu^T$$

This unit vector u is the Householder vector & we can compute it in MATLAB (you'll do this on HW)