LECTURE 17

Compare CGS with MGS; both to QR (then) Householder triangularization intro

Q. Consider the matrix

$$A = \begin{bmatrix} 1 & 1 & 1 \\ \varepsilon & & \\ & \varepsilon & \\ & & \varepsilon \end{bmatrix}$$

Run the classical GS algorithm until you have q_1 and q_2 (and r_{11}, r_{12}, r_{22}).

<u>CGS Algorithm</u>:

$a_1 = [1 \;\; \varepsilon \;\; 0 \;\; 0]^T$

$a_2 = [1 \;\; 0 \;\; \varepsilon \;\; 0]^T$

$a_3 = [1 \;\; 0 \;\; 0 \;\; \varepsilon]^T$

<u>Step j=1</u>

$v_1^{(1)} = a_1$

$r_{11} = \| v_1^{(1)} \| = \sqrt{1^2 + \varepsilon^2} \approx 1$ $\qquad 1 \gg \varepsilon^2$

$q_1 = \dfrac{v_1^{(1)}}{r_{11}} = [1 \;\; \varepsilon \;\; 0 \;\; 0]^T$

<u>Step j=2</u>

$v_2^{(1)} = a_2$

$r_{12} = q_1^T a_2 = [1 \;\; \varepsilon \;\; 0 \;\; 0] \cdot [1 \;\; 0 \;\; \varepsilon \;\; 0] = 1$

$v_2^{(2)} = v_2^{(1)} - q_1 = [1 \;\; 0 \;\; \varepsilon \;\; 0]^T - [1 \;\; \varepsilon \;\; 0 \;\; 0]^T = [0 \;\; -\varepsilon \;\; \varepsilon \;\; 0]^T$

$r_{22} = \| v_2^{(2)} \| = \sqrt{2\varepsilon^2} = \sqrt{2}\,\varepsilon$

$q_2 = \dfrac{v_2^{(2)}}{r_{22}} = \dfrac{1}{\sqrt{2}\,\varepsilon} [0 \;\; -\varepsilon \;\; \varepsilon \;\; 0]^T = [0 \;\; -1/\sqrt{2} \;\; 1/\sqrt{2} \;\; 0]^T$

Step $j=3$

$v_3^{(1)} = a_3$

$r_{13} = q_1^T a_3 = [1\ \varepsilon\ 0\ 0]\cdot[1\ 0\ 0\ \varepsilon] = 1$

$v_3^{(2)} = v_3^{(1)} - r_{13} q_1 = [1\ 0\ 0\ \varepsilon]^T - [1\ \varepsilon\ 0\ 0]^T = [0\ -\varepsilon\ 0\ \varepsilon]^T$

$r_{23} = q_2^T a_3 = [0\ \ -1/\sqrt{2}\ \ 1/\sqrt{2}\ \ 0]\cdot[1\ 0\ 0\ \varepsilon] = 0$

$v_3^{(3)} = v_3^{(2)} - 0 = [0\ -\varepsilon\ 0\ \varepsilon]^T$

$r_{33} = \| v_3^{(3)} \| = \sqrt{2\varepsilon^2} = \sqrt{2}\,\varepsilon$

$q_3 = \dfrac{v_3^{(3)}}{r_{33}} = \dfrac{1}{\sqrt{2}\,\varepsilon}[0\ -\varepsilon\ 0\ \varepsilon]^T = [0\ \ -1/\sqrt{2}\ \ 0\ \ 1/\sqrt{2}]^T$

Output:

$$\hat{Q}_c = \begin{bmatrix} 1 & 0 & 0 \\ \varepsilon & -1/\sqrt{2} & -1/\sqrt{2} \\ 0 & 1/\sqrt{2} & 0 \\ 0 & 0 & 1/\sqrt{2} \end{bmatrix}$$

Not stored: $\hat{R} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & \sqrt{2}\,\varepsilon & 0 \\ 0 & 0 & \sqrt{2}\,\varepsilon \end{bmatrix}$

How much loss of orthogonality?

$$\hat{Q}_c^T \hat{Q}_c = \begin{bmatrix} 1 & \varepsilon & 0 & 0 \\ 0 & -1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 0 & -1/\sqrt{2} & 0 & 1/\sqrt{2} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ \varepsilon & -1/\sqrt{2} & -1/\sqrt{2} \\ 0 & 1/\sqrt{2} & 0 \\ 0 & 0 & 1/\sqrt{2} \end{bmatrix} = \begin{bmatrix} 1+\varepsilon^2 & -\varepsilon/\sqrt{2} & -\varepsilon/\sqrt{2} \\ -\varepsilon/\sqrt{2} & 1 & 1/2 \\ -\varepsilon/\sqrt{2} & 1/2 & 1 \end{bmatrix}$$

$$\approx \begin{bmatrix} 1 & -\varepsilon/\sqrt{2} & -\varepsilon/\sqrt{2} \\ -\varepsilon/\sqrt{2} & 1 & 1/2 \\ -\varepsilon/\sqrt{2} & 1/2 & 1 \end{bmatrix}$$

The $\pm\varepsilon/\sqrt{2}$ are okay, but the $1/2$'s are bad.

$$\left(\hat{Q}\hat{R} = \begin{bmatrix} 1 & 0 & 0 \\ \varepsilon & -1/\sqrt{2} & -1/\sqrt{2} \\ 0 & 1/\sqrt{2} & 0 \\ 0 & 0 & 1/\sqrt{2} \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 0 & \sqrt{2}\,\varepsilon & 0 \\ 0 & 0 & \sqrt{2}\,\varepsilon \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ \varepsilon & \varepsilon-\varepsilon & \varepsilon-\varepsilon \\ 0 & \varepsilon & 0 \\ 0 & 0 & \varepsilon \end{bmatrix} = A \text{ indeed} \right)$$

MGS Algorithm

$a_1 = [1 \;\; \varepsilon \;\; 0 \;\; 0]^T$

$a_2 = [1 \;\; 0 \;\; \varepsilon \;\; 0]^T$

$a_3 = [1 \;\; 0 \;\; 0 \;\; \varepsilon]^T$

Setup

$v_1^{(1)} = a_1$

$v_2^{(1)} = a_2$

$v_3^{(1)} = a_3$

Step $i=1$

$$r_{11} = \|v_1^{(1)}\| = \sqrt{1^2 + \varepsilon^2} \approx 1$$

$$q_1 = \frac{v_1^{(1)}}{r_{11}} = [1 \;\; \varepsilon \;\; 0 \;\; 0]^T$$

$$r_{12} = q_1^T v_2^{(1)} = [1 \;\; \varepsilon \;\; 0 \;\; 0] \cdot [1 \;\; 0 \;\; \varepsilon \;\; 0] = 1$$

$$v_2^{(2)} = v_2^{(1)} - q_1 = [1 \;\; 0 \;\; \varepsilon \;\; 0]^T - [1 \;\; \varepsilon \;\; 0 \;\; 0]^T = [0 \;\; -\varepsilon \;\; \varepsilon \;\; 0]^T$$

$$r_{13} = q_1^T v_3^{(1)} = [1 \;\; \varepsilon \;\; 0 \;\; 0] \cdot [1 \;\; 0 \;\; 0 \;\; \varepsilon] = 1$$

$$v_3^{(2)} = v_3^{(1)} - q_1 = [1 \;\; 0 \;\; 0 \;\; \varepsilon]^T - [1 \;\; \varepsilon \;\; 0 \;\; 0]^T = [0 \;\; -\varepsilon \;\; 0 \;\; \varepsilon]^T$$

Step $i=2$

$$r_{22} = \|v_2^{(2)}\| = \sqrt{\varepsilon^2 + \varepsilon^2} = \sqrt{2}\,\varepsilon$$

$$q_2 = \frac{v_2^{(2)}}{r_{22}} = \frac{1}{\sqrt{2}\,\varepsilon} [0 \;\; -\varepsilon \;\; \varepsilon \;\; 0]^T = [0 \;\; -1/\sqrt{2} \;\; 1/\sqrt{2} \;\; 0]^T$$

$$r_{23} = q_2^T v_3^{(2)} = [0 \;\; -1/\sqrt{2} \;\; 1/\sqrt{2} \;\; 0] \cdot [0 \;\; -\varepsilon \;\; 0 \;\; \varepsilon] = \varepsilon/\sqrt{2}$$

$$v_3^{(3)} = v_3^{(2)} - r_{23} q_2 = [0 \;\; -\varepsilon \;\; 0 \;\; \varepsilon]^T - \varepsilon/\sqrt{2}\, [0 \;\; -1/\sqrt{2} \;\; 1/\sqrt{2} \;\; 0]^T$$

$$= [0 \;\; -\varepsilon \;\; 0 \;\; \varepsilon]^T + [0 \;\; \varepsilon/2 \;\; -\varepsilon/2 \;\; 0]^T$$

$$= [0 \;\; -\varepsilon/2 \;\; -\varepsilon/2 \;\; \varepsilon]^T$$

Step $i=3$

$$r_{33} = \|v_3^{(3)}\| = \sqrt{\varepsilon^2/4 + \varepsilon^2/4 + \varepsilon^2} = \sqrt{\frac{6\varepsilon^2}{4}} = \frac{\sqrt{6}\,\varepsilon}{2}$$

$$q_3 = \frac{v_3^{(3)}}{r_{33}} = \frac{2}{\sqrt{6}\,\varepsilon} [0 \;\; -\varepsilon/2 \;\; -\varepsilon/2 \;\; \varepsilon] = [0 \;\; -1/\sqrt{6} \;\; -1/\sqrt{6} \;\; 2/\sqrt{6}]$$

Output:

$$\hat{Q}_M = \begin{bmatrix} 1 & 0 & 0 \\ \varepsilon & -1/\sqrt{2} & -1/\sqrt{6} \\ 0 & 1/\sqrt{2} & -1/\sqrt{6} \\ 0 & 0 & 2/\sqrt{6} \end{bmatrix}$$

Still not actually orthogonal, but close enough:

$$\hat{Q}_M^T \hat{Q}_M = \begin{bmatrix} 1 & \varepsilon & 0 & 0 \\ 0 & -1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 0 & -1/\sqrt{6} & -1/\sqrt{6} & 2/\sqrt{6} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ \varepsilon & -1/\sqrt{2} & -1/\sqrt{6} \\ 0 & 1/\sqrt{2} & -1/\sqrt{6} \\ 0 & 0 & 2/\sqrt{6} \end{bmatrix}$$

$$= \begin{bmatrix} 1+\varepsilon^2 & -\varepsilon/\sqrt{2} & -\varepsilon/\sqrt{6} \\ -\varepsilon/\sqrt{2} & 1 & 0 \\ -\varepsilon/\sqrt{6} & 0 & 1 \end{bmatrix} \approx \begin{bmatrix} 1 & -\varepsilon/\sqrt{2} & -\varepsilon/\sqrt{6} \\ -\varepsilon/\sqrt{2} & 1 & 0 \\ -\varepsilon/\sqrt{6} & 0 & 1 \end{bmatrix}$$

from CGS: $\hat{Q}_c^T \hat{Q}_c = \begin{bmatrix} 1 & -\varepsilon/\sqrt{2} & -\varepsilon/\sqrt{2} \\ -\varepsilon/\sqrt{2} & 1 & 1/2 \\ -\varepsilon/\sqrt{2} & 1/2 & 1 \end{bmatrix}$