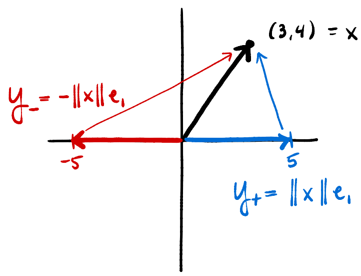
Lecture 18

Q. Write down the Householder reflection matrix that takes the vector $x = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$ to a vector y in the direction of $e_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

Find y. * Choose a vector y! *

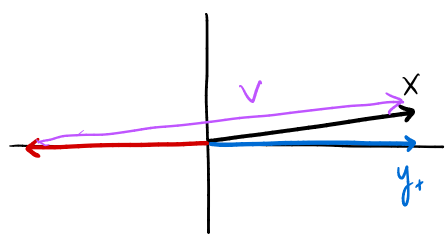
$\|x\| = \sqrt{3^2+4^2} = 5$. There are two vectors: $y = \begin{bmatrix} 5 \\ 0 \end{bmatrix}$ or $\begin{bmatrix} -5 \\ 0 \end{bmatrix}$



Recall: we want $v = x - y$, $u = \frac{v}{\|v\|}$.

Then $H = I - 2uu^T$.

In general, we want to vector that is "most different" from x, because of situations like:



here $x \approx +\|x\|e_1 = y_+$

so $v = x - y_+ \approx 0$ (Error!)

Best to always choose $v = x + \underbrace{\mathrm{sgn}(x_1)}_{\text{sign of } e_1 \text{ component of } x} \|x\| e_1$:

$$v = x - y_- = x - (-\|x\|e_1) = \begin{bmatrix} 3 \\ 4 \end{bmatrix} + \begin{bmatrix} 5 \\ 0 \end{bmatrix} = \begin{bmatrix} 8 \\ 4 \end{bmatrix} = 4\begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

$$\|v\| = 4\sqrt{4+1} = 4\sqrt{5}$$

$$u = \frac{v}{\|v\|} = \frac{1}{4\sqrt{5}} \cdot 4\begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 2/\sqrt{5} \\ 1/\sqrt{5} \end{bmatrix}$$

$$H = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - 2\begin{bmatrix} 2/\sqrt{5} \\ 1/\sqrt{5} \end{bmatrix}\begin{bmatrix} 2/\sqrt{5} & 1/\sqrt{5} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 8/5 & 4/5 \\ 4/5 & 2/5 \end{bmatrix} = \begin{bmatrix} -3/5 & -4/5 \\ -4/5 & 3/5 \end{bmatrix}$$

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H is an orthogonal matrix. $y = Hx$

MGS as triangular orthogonalization:

Lets write the algorithm as matrix operations:

Step $i=1$

$$\underbrace{\begin{bmatrix} v_1^{(1)} & v_2^{(1)} & \cdots & v_n^{(1)} \end{bmatrix}}_{=A} \underbrace{\left[\begin{array}{c|ccc} \frac{1}{r_{11}} & -\frac{r_{12}}{r_{11}} & \cdots & -\frac{r_{1n}}{r_{11}} \\ \hline 0 & & I_{n-1} & \\ & & & \end{array}\right]}_{\substack{\text{triangular operation} \\ \text{"}R_1\text{"}}} = \underbrace{\begin{bmatrix} q_1 & v_2^{(2)} & \cdots & v_n^{(2)} \end{bmatrix}}_{=AR_1}$$

Step $i=2$

$$\underbrace{\begin{bmatrix} q_1 & v_2^{(2)} & \cdots & v_n^{(2)} \end{bmatrix}}_{AR_1} \underbrace{\left[\begin{array}{c|c|ccc} 1 & 0 & & \cdots & 0 \\ \hline 0 & \frac{1}{r_{22}} & -\frac{r_{23}}{r_{22}} & \cdots & -\frac{r_{2n}}{r_{22}} \\ \cline{2-5} \vdots & 0 & & I_{n-2} & \\ 0 & & & & \end{array}\right]}_{R_2} = \underbrace{\begin{bmatrix} q_1 & q_2 & v_3^{(3)} & \cdots & v_n^{(3)} \end{bmatrix}}_{AR_2R_3}$$

$$\cdots \quad AR_1R_2\cdots R_n = \hat{Q} = [q_1 \cdots q_n]$$

$$R_n = \begin{bmatrix} 1 & & & 0 \\ & 1 & & \\ & & \ddots & \\ 0 & & 1 & \frac{1}{r_{nn}} \end{bmatrix}$$

* Note that each R_i is invertible!
(full rank, square matrix)

Let $\hat{R} = (R_1R_2\cdots R_n)^{-1}$. Then $A = \hat{Q}\hat{R}$ indeed.

Householder Triangularization is orthogonal triangularization:

$$\underset{A}{\begin{bmatrix} \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet \end{bmatrix}} \xrightarrow{Q_1} \underset{Q_1A}{\begin{bmatrix} \triangle & \triangle & \triangle \\ 0 & \triangle & \triangle \\ 0 & \triangle & \triangle \\ 0 & \triangle & \triangle \end{bmatrix}} \xrightarrow{Q_2} \underset{Q_2Q_1A}{\begin{bmatrix} \triangle & \triangle & \triangle \\ 0 & \square & \square \\ 0 & 0 & \square \\ 0 & 0 & \square \end{bmatrix}} \xrightarrow{Q_3} \underset{Q_3Q_2Q_1A \,=\, R}{\begin{bmatrix} \triangle & \triangle & \triangle \\ 0 & \square & \square \\ 0 & 0 & \ast \\ 0 & 0 & 0 \end{bmatrix}} = R$$

(full R!)

Let $Q^T = Q_3Q_2Q_1$. Then $Q^TA = R \Rightarrow A = QR$ (full QR decomp)

<u>Algorithm</u> (Householder QR decomposition) $\qquad A \in \mathbb{R}^{m \times n}$

for $k = 1 : n$

$x = A(k:m, k)$ $\qquad$ eg. $n=3$, $m=4$, $k=2$:

$$\begin{array}{c} \begin{bmatrix} \Delta & \Delta & \Delta \\ 0 & \boxed{\Delta} & \Delta \\ 0 & \boxed{\Delta} & \Delta \\ 0 & \boxed{\Delta} & \Delta \end{bmatrix} \\ \uparrow x \end{array}$$

$v_k = \text{sgn}(x_1)\|x\| e_1 + x$

in standard basis for $\mathbb{R}^{m-k+1}$

$= \mathbb{R}_k \times \mathbb{R}_{k+1} \times \cdots \times \mathbb{R}_m$

$v_k = \dfrac{v_k}{\|v_k\|}$ $\qquad$ normalized ("u_k")

$$A(k:m, k:n) = \underbrace{A(k:m, k:n) - 2 v_k v_k^T A(k:m, k:n)}_{(I - 2u_k u_k^T)(A(k:m, k:n))}$$

<u>Output</u> R, whose top square $\hat{R}$ is upper triangular

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You might not wish to store the Q_i if not needed:

eg. $\quad Ax = b$ $\quad$ (determined or overdetermined)

If you have $Q^T A = R$ as linear transformation,

then $\qquad Rx = Q^T A x$ $\qquad$ ($= Q^T b$ if $Ax = b$ is determined)

So if we want $\quad \min_x \|b - Ax\|^2 = \min_x \|Q^T b - Q^T A x\|^2$

$= \min_x \|Rx - Q^T b\|$

so we didn't need to store Q^T.

<u>Note</u> You could compute $Q^T b$ by running the algorithm

for $k = 1:n$ $\quad$ on $\quad [a_1 \; a_2 \; \dots \; a_n \mid b]$

↑ <u>not</u> $n+1$!

& not store Q^T.

Numerical stability example

$$A = \begin{bmatrix} 1 & 1 & 1 \\ \varepsilon & & \\ & \varepsilon & \\ & & \varepsilon \end{bmatrix} \qquad \begin{aligned} a_1 &= [1 \quad \varepsilon \quad 0 \quad 0]^T \\ a_2 &= [1 \quad 0 \quad \varepsilon \quad 0]^T \\ a_3 &= [1 \quad 0 \quad 0 \quad \varepsilon]^T \end{aligned}$$

Using Householder reflections, we get

$$\underbrace{\underbrace{\begin{bmatrix} 1 & & & \\ & 1 & & \\ & & -1/\sqrt{3} & \sqrt{2}/\sqrt{3} \\ & & \sqrt{2}/\sqrt{3} & 1/\sqrt{3} \end{bmatrix}}_{Q_3} \underbrace{\begin{bmatrix} 1 & & & \\ & -1/\sqrt{2} & 1/\sqrt{2} & \\ & 1/\sqrt{2} & 1/\sqrt{2} & \\ & & & 1 \end{bmatrix}}_{Q_2} \underbrace{\begin{bmatrix} -1 & -\varepsilon & & \\ -\varepsilon & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix}}_{Q_1}}_{Q^T} \underbrace{\begin{bmatrix} 1 & 1 & 1 \\ \varepsilon & & \\ & \varepsilon & \\ & & \varepsilon \end{bmatrix}}_{A} = \underbrace{\begin{bmatrix} -1 & -1 & -1 \\ & 2\varepsilon/\sqrt{2} & \varepsilon/\sqrt{2} \\ & & \left(1/\sqrt{3} + \sqrt{2}/\sqrt{3}\right)\varepsilon \\ & & \end{bmatrix}}_{R}$$

$\Longrightarrow A = QR$

So $Q^T = Q_3 Q_2 Q_1$ and $Q = (Q^T)^T = Q_1^T Q_2^T Q_3^T$

and $Q^T Q = Q_3 Q_2 \underbrace{Q_1 \, Q_1^T} Q_2^T Q_3^T$

there are some epsilons here:

$$Q_1^T Q = \begin{bmatrix} -1 & -\varepsilon \\ -\varepsilon & 1 \end{bmatrix} \begin{bmatrix} -1 & -\varepsilon \\ -\varepsilon & 1 \end{bmatrix} = \begin{bmatrix} 1+\varepsilon^2 & 0 \\ 0 & 1+\varepsilon^2 \end{bmatrix} \approx I_2 \,!$$

Turns out $Q^T Q \approx I_4$; only multiples of ε^2 appear!