# MIDTERM EXAM SOLUTIONS

① (a)

$$\text{terms in alphabetical order}\left\{\begin{array}{l} \text{Americano} \\ \text{Caramel} \\ \text{Decaf} \\ \text{Iced} \\ \text{Latte} \\ \text{Soy} \\ \text{Vanilla} \end{array}\right. \overbrace{\begin{array}{ccc} D_1 & D_2 & D_3 \end{array}}^{\text{documents}} \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix} = A.$$

(b)

$$q = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

(c) $$\cos\theta(D_1, q) = \frac{D_1^T q}{\|D_1\|_2 \|q\|_2}$$

$$D_1^T q = D_1 \cdot q = [0\ 1\ 0\ 1\ 1\ 0\ 0] \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} = 2$$

$$\|D_1\|_2 = \sqrt{3}, \quad \|q\|_2 = \sqrt{2}$$

$$\Rightarrow \quad \cos\theta(D_1, q) = \frac{D_1^T q}{\|D_1\|_2 \|q\|_2} = \frac{2}{\sqrt{3} \cdot \sqrt{2}}$$

② (a) $\|a_{\cdot 1}\|_1 = |-2| + |1| = 3$ $\qquad \|a_{\cdot 2}\| = |1| + |3| = 4.$

(b) $$pq^T = \begin{bmatrix} 5 \\ 4 \end{bmatrix} \begin{bmatrix} -1 & 1 \end{bmatrix} = \begin{bmatrix} -5 & 5 \\ -4 & 4 \end{bmatrix}$$

This is an outer product of matrices and is therefore rank 1.

(c) Since $\det(A) = -6 - 1 = -7 \neq 0$, $A$ is invertible, and so its column span is all of $\mathbb{R}^2$. Therefore $p, q,$ and $r$ are <u>all</u> in the column span of $A$.

(d) $\|A\|_F = \sqrt{4 + 1 + 1 + 9} = \sqrt{15}$

(e) $$\|A\|_2 = \left(\max\left\{\lambda_i(A^TA)\right\}\right)^{1/2}$$

$$A^TA = \begin{bmatrix} -2 & 1 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} -2 & 1 \\ 1 & 3 \end{bmatrix} = \begin{bmatrix} 5 & 1 \\ 1 & 10 \end{bmatrix}$$

$$A^TA - \lambda I = \begin{bmatrix} 5 & 1 \\ 1 & 10 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} = \begin{bmatrix} 5-\lambda & 1 \\ 1 & 10-\lambda \end{bmatrix}$$

$$\det(A^TA - \lambda I) = (5-\lambda)(10-\lambda) - 1 = 50 - 15\lambda + \lambda^2 - 1 = \lambda^2 - 15\lambda + 49$$

$$\lambda_\pm = \frac{15 \pm \sqrt{15^2 - 4(49)}}{2} \qquad (\lambda_+ > \lambda_-)$$

$$\Rightarrow \|A\|_2 = \sqrt{\lambda_+} = \sqrt{\frac{15 \pm \sqrt{15^2 - 4(49)}}{2}}$$

(3) (a)

$$\begin{array}{cc} 1 & x \\ \end{array}$$

$$\begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} c_0 \\ c_1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 4 \end{bmatrix}$$

(b) $A^T A c = A^T y$

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} c_0 \\ c_1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 4 \end{bmatrix}$$

$$\begin{bmatrix} 3 & 3 \\ 3 & 5 \end{bmatrix} \begin{bmatrix} c_0 \\ c_1 \end{bmatrix} = \begin{bmatrix} 6 \\ 9 \end{bmatrix}$$

Solving using substitution:

$$\Rightarrow \begin{cases} 3c_0 + 3c_1 = 6 \\ 3c_0 + 5c_1 = 9 \end{cases}$$

$$c_0 + c_1 = 2 \Rightarrow c_1 = 2 - c_0$$

$$3c_0 + 5(2 - c_0) = 3c_0 + 10 - 5c_0$$
$$= -2c_0 + 10 = 9$$

$$\Rightarrow c_0 = \frac{1}{2}, \quad c_1 = 2 - \frac{1}{2} = \frac{3}{2}.$$

Solving using matrix inverse.

$$\begin{pmatrix} 3 & 3 \\ 3 & 5 \end{pmatrix}^{-1} = \frac{1}{|15-9|} \begin{pmatrix} 5 & -3 \\ -3 & 3 \end{pmatrix} = \frac{1}{6} \begin{pmatrix} 5 & -3 \\ -3 & 3 \end{pmatrix}$$

$$\begin{bmatrix} c_0 \\ c_1 \end{bmatrix} = \begin{pmatrix} 3 & 3 \\ 3 & 5 \end{pmatrix}^{-1} \begin{pmatrix} 6 \\ 9 \end{pmatrix} = \frac{1}{6} \begin{pmatrix} 5 & -3 \\ -3 & 3 \end{pmatrix} \begin{pmatrix} 6 \\ 9 \end{pmatrix}$$

$$= \frac{1}{6} \begin{pmatrix} 30 - 27 \\ -18 + 27 \end{pmatrix} = \frac{1}{6} \begin{pmatrix} 3 \\ 9 \end{pmatrix} = \begin{pmatrix} 1/2 \\ 3/2 \end{pmatrix}$$

Best fit line: $y = \frac{1}{2} + \frac{3}{2}x$