MAT167 HW01

Due 4/13/23 at 11:59 pm on Gradescope

Instructions You may handwrite or type up your homework solution. Regardless, your solutions **must be neat**. If your solutions are incoherent, illegible, or difficult to read, you will lose style points during grading.

- You must justify your answers, i.e. show sufficient steps so that the grader can see that you understand the algorithm. Aside from calculations, your solutions should be written in **full sentences.**
- Create PDFs for your solutions. If you have handwritten work, you need to scan these (e.g. using a scanner app); you should export any code or computer output to a PDF as well. Finally, combine these PDFs into **one single PDF**.
- Submit the **one single PDF** to Gradescope. If you decide to change your solution, you can update your PDF submission as many times as you'd like, as long as the deadline hasn't passed.

Some useful MATLAB commands

Example	Action
`x = [1, 2 ,3]`	assigns the row vector $[1, 2, 3]$ to a variable x
`y = x'`	assigns to y the column vector $x^\top$
`A*B`	computes the product of the matrices A and B
`A(:,2)*x`	computes the product of (the column vector) $a_{\bullet 2}$ and (a row vector) x, as matrices

Exercise 1

This is a simple MATLAB exercise related to the "music compression" example from the first day of class. First, go to the Announcements section of the class website and follow the instructions for how to access MATLAB. Then, follow the steps below. Only some of them have outputs that you'll need to report on your homework; you need only submit the requested figures and responses; you should *not* submit your MATLAB code.

(a) Download the data file `hw01.mat`, which contains

- a signal `x` consisting of 8 points, stored as a vector `x` $\in \mathbb{R}^8$, and
- a matrix `U`, consisting of 8 basis vectors `U(:,k)` for $1 \leq k \leq 8$ (i.e. $k \in [8]$ in my notation).

Save this to your working directory. In the MATLAB terminal, type

```
>> load hw01;
```

to load the data into your MATLAB session. Then, draw the signal `x` in the data file using the following commands:

```
>> figure(1);
>> stem(x); hold on; plot(x); grid;
```

(b) In a different figure window, draw the 8 basis vectors stored as column vectors of the matrix U as follows:

```
>> figure(2);
>> for k=1:8
     subplot(8,1,k);
     stem(U(:,k)); axis([0 9 -0.5 0.5]); axis off; hold on;
   end
>> for k=1:8
     subplot(8,1,k);
     plot(U(:,k));
   end
```

You may need to see the details of these 8 plots by enlarging the window to a full screen. **Save Figure 2 and include it in your HW submission.**

(c) Compute the expansion coefficients (i.e., the weights of the linear combination) of `x` with respect to the basis vectors `U(:,1), ..., U(:,8)` using

```
>> a=U'*x;
```

(d) Check the values of the entries of the coefficient vector `a` and create a new vector `a2` of length 8 whose only nonzero entries are the two largest entries of `a` in terms of their absolute values.

(e) Construct an approximation `x2` of `x` using `a2`. Then, plot `x2` over Figure 1 as follows:

```
>> figure(1); stem(x2,'r*'); plot(x2,'r');
```

(f) Now, instead of `a2`, let's construct `a4` of length 8 whose only nonzero entries are the four entries of `a` with largest absolute values.

(g) Construct an approximation `x4` of `x` using `a4`. Then, plot `x4` over Figure 1 as follows (note how the color is different from that of `x2`):

```
>> figure(1); stem(x4,'gx'); plot(x4,'g');
```

Now save Figure 1 and include it in your HW submission.

(h) Now consider `x8`, which is just a full reconstruction without throwing out any coefficients, i.e.,

```
>> x8=U*a;
```

Finally, compute the relative error of `x8` by

```
>> sqrt(sum((x-x8).^2)/sum(x.^2))
```

and **report the result**. Note that there will be a nonzero error due to floating-point approximations!

Similarly compute the relative error of `x4` and `x2`, and **report the results**.

Exercise 2

Consider the following set of terms (words) and documents (book titles):

	Terms		Documents
T1:	Book (Handbook, BOOK)	D1:	The Princeton Companion to Mathematics
T2:	Equation (Equations)	D2:	NIST Handbook of Mathematical Functions
T3:	Function (Functions)	D3:	Table of Integrals, Series, and Products
T4:	Integral (Integrals)	D4:	Linear Integral Equations
T5:	Linear	D5:	Proofs from THE BOOK
T6:	Mathematics (Mathematical)	D6:	The Book of Numbers
T7:	Number (Numbers)	D7:	Number Theory in Science and Communication
T8:	Series	D8:	Green's Functions and Boundary Value Problems
		D9:	Discourse on Fourier Series
		D10:	Basic Linear Partial Differential Equations
		D11:	Mathematical Physics, An Advanced Course

(a) Construct the 8×11 term-document matrix.

(b) Suppose we want to query "Integral Equation." Construct the query vector $\mathbf{q}$.

(c) We can use the L_1 norm to measure the distance between two vectors. Write down a matrix calculation that (i.e. equation) that outputs a vector $\mathbf{d}$ such that d_i is the distance between $\mathbf{q}$ and document Di.

(d) Use MATLAB to compute the vector $\mathbf{d}$ above. Then, report back on which three documents are closest to "Integral Equation" with respect to the L_1 distance measure.

(e) Recall that we can also use the cosine of the angle between two vectors as a measure of distance between them. Write down a matrix calculation (i.e. equation) that outputs a vector $\mathbf{c}$ such that c_i is the distance between $\mathbf{q}$ and document Di.

(f) Use MATLAB to compute the vector $\mathbf{c}$ above. Then, report back on which three documents are closest to "Integral Equation" with respect to the cosine-of-the-angle distance measure.

(g) Suppose that instead of book titles, our documents were variable length. (For example, we could consider full articles D1', D2', etc.) If we compared the query term "Integral Equation" to the longer documents, how would might our L_1-distance ranking differ from our cosine-of-the-angle-distance ranking?

Exercise 3

(This is a review problem.) Consider the following matrix:

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 5 & 7 \\ 0 & 0 & 1 & 6 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

(a) View this matrix as a linear transformation between two vector spaces. Identify the domain and the codomain of this linear transformation.

(b) Identify the image of this linear transformation.

(c) Write a basis for the image of this linear transformation.

(d) Identify the kernel of this linear transformation.

(e) Write a basis for the kernel of this linear transformation.

(f) What is the column space of this linear transformation?

(g) What is the row space of this linear transformation?

(h) What is the rank of the matrix A?

Exercise 4

(This is a review problem.) Use the Gram-Schmidt Process to construct an orthonormal basis for $\mathbb{R}^4$ starting with the following set of vectors.

$$\mathbf{x}_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \qquad \mathbf{x}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ -1 \end{bmatrix}, \qquad \mathbf{x}_3 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \qquad \mathbf{x}_4 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$