MAT167 HW02

Due 4/20/23 at 11:59 pm on Gradescope

Instructions You may handwrite or type up your homework solution. Regardless, your solutions **must be neat**. If your solutions are incoherent, illegible, or difficult to read, you will lose style points during grading.

- You must justify your answers, i.e. show sufficient steps so that the grader can see that you understand the algorithm. Aside from calculations, your solutions should be written in **full sentences.**
- Create PDFs for your solutions. If you have handwritten work, you need to scan these (e.g. using a scanner app); you should export any code or computer output to a PDF as well. Finally, combine these PDFs into **one single PDF**.
- Submit the **one single PDF** to Gradescope. If you decide to change your solution, you can update your PDF submission as many times as you'd like, as long as the deadline hasn't passed.

This homework covers material from Chapter 2 in the book. Below, if a problem says "prove that X", this means you should explain why the statement X is true.

Exercise 1

Consider a matrix $A \in \mathbb{R}^{m\times n}$. Prove that the column rank of A is the same as the row rank of A.

Exercise 2

Let $A \in \mathbb{R}^{n\times n}$ be a square matrix. Prove that if the nullspace $null(A) = \{0\}$, then A is invertible.

Exercise 3

Let S denote set of points in $\mathbb{R}^2$ where the Euclidean norm is 1:

$$S = \{x \in \mathbb{R}^2 \ : \ \|x\|_2 = 1\}.$$

What is the minimum value of $\|x\|_1$ for $x \in S$, and at which points x does this minimum value occur?

Hint: S is the "unit circle" we're used to from calculus. We can parametrize the points in S using an angle θ between 0 and 2π. What is the vector (i.e. rectangular coordinates) for the point corresponding to angle θ?

Exercise 4

Let $\|\cdot\|$ denote both a norm on $\mathbb{R}^m$ and also the induced matrix norm on $\mathbb{R}^{m\times m}$.

Let $\rho(A)$ be the *spectral radius* of A

$$\rho(A) := \max_{1\leq i\leq m} |\lambda_i(A)|$$

where the λ_i are the eigenvalues of A. Prove that $\rho(A) \leq \|A\|$.

Note that the norm in this problem could be any norm; you are not allowed to choose which one.

Exercise 5

In this problem, you will investigate matrix norms using MATLAB.

(a) Assign the matrix

$$A = \begin{bmatrix} 1 & 2 \\ 0 & 2 \\ 1 & 3 \end{bmatrix}$$

to the variable `A` in MATLAB. Then, compute the 2-norm by using the `norm` function, and report the result in a `long` format (i.e. 16 digits) using the commands

```
>> format long
>> norm(A)
```

(b) Compute the 2-norm explicitly by first computing the largest eigenvalue of $A^\top A$ using the `eig` function:

```
>> sqrt(max(eig(A'*A)))
```

Compare the result with part (a). What is the relative error between the norms computed in part (a) and part (b)?

(c) Compute the 1-norm, ∞-norm, and Frobenius norm of `A` by hand, using the formulas from class. Then, refer to

```
>> help norm
```

to discover how to compute these norms (the 1-norm, ∞-norm, and Frobenius norm) using MATLAB.

Compare your hand-computed results with those computed in MATLAB.

(d) Load the MATLAB data file `hw01.mat` from HW01 (still available on the class website). Compute the coefficient vector a as you did last time. Using the `norm` function, compute $\|x\|_p$ and $\|a\|_p$ for each $p \in \{1, 2, \infty\}$. Report the results of these (six) computations. For which value of p did you get $\|x\|_p = \|a\|_p$?

(e) Using the `norm` function, compute the associated matrix norms $\|U\|_p$ for $p \in \{1, 2, \infty\}$, as well as $\|U\|_F$, and report the results.