MAT167 HW03

Due 4/27/23 at 11:59 pm on Gradescope

Instructions You may handwrite or type up your homework solution. Regardless, your solutions **must be neat**. If your solutions are incoherent, illegible, or difficult to read, you will lose style points during grading.

- You must justify your answers, i.e. show sufficient steps so that the grader can see that you understand the algorithm. Aside from calculations, your solutions should be written in **full sentences.**
- Create PDFs for your solutions. If you have handwritten work, you need to scan these (e.g. using a scanner app); you should export any code or computer output to a PDF as well. Finally, combine these PDFs into **one single PDF**.
- Submit the **one single PDF** to Gradescope. If you decide to change your solution, you can update your PDF submission as many times as you'd like, as long as the deadline hasn't passed.

Covered material This homework covers material from §3.3 and §3.6 in the book. You should read these sections before you start the homework.

Midterm Exam Information

The midterm will be in class on Friday, April 28, and will be a traditional paper-and-pencil exam. You are not allowed to use any electronic device during the exam; all answers will be submitted either as explanations or by-hand calculations yielding an expression that could be entered into a calculator or computer.

So far in class we have covered the following material from the book:

- Chapter 1: an intro to the themes of the course, as well as the notations and conventions we are using in the book
- Chapter 2: what matrix-matrix and matrix-vector multiplications represent (e.g. coefficients in a linear combination, linear system, etc.), how to measure vectors and matrices, linear independence, bases, rank, nullity, image, nullspace, etc.
- §3.3 and §3.6 in Chapter 3: linear least squares problems, normal equations, condition number

Some of you have asked for a list of topics from the prerequisite classes that are particularly important to the material covered on the midterm:

- solving linear systems

- LU decomposition
- rank, span, nullity, image, kernel, domain, codomain
- linear combinations, linear independence / dependence
- basis vectors, change of basis
- column space and row space
- orthogonal vectors
- dot product and outer product
- eigenvalues, eigenvectors

This is obviously not a complete list (e.g. I hope I don't have to write "transpose" in the list), but hopefully this will be helpful as a guide.

Exercise 1

Using MATLAB, complete the following procedure:

(a) Download the data file hw03.mat to your working directory, name it as hw03.mat. Then, load it into your MATLAB session. Check what variables (i.e., arrays) are defined in this data file by running:

```
>> whos
```

(b) Plot the data:

```
>> plot(x,y); grid;
```

(c) Create the *Vandermonde matrix* for polynomials of degree 1 (i.e., lines) by:

```
>> A=[x.^0 x.^1];
```

(d) Compute the least squares line over the given data:

```
>> sol = inv(A'*A)*A'*y;
```

Then, overlay the least squares line over the current plot:

```
>> hold on; plot(x, sol(1)+sol(2)*x, '--');
```

Add title and axis labels:

```
>> title('Least squares line fit'); xlabel('x'); ylabel('y');
```

Now save this plot and submit it as part of your homework solution.

Exercise 2

We want to prove the following statement: *If $A \in \mathbb{R}^{m\times n}$ is full rank, then $A^\top A$ is nonsingular (i.e., invertible).* Let's prove this step-by-step.

(a) Let $\mathbf{x}, \mathbf{z} \in \mathbb{R}^n$, $\mathbf{b} \in \mathbb{R}^m$, $\alpha \in \mathbb{R}$, and consider the following:

$$\|A(\mathbf{x} + \alpha\mathbf{z}) - \mathbf{b}\|_2^2.$$

Write this quantity as a quadratic polynomial in terms of α.

(b) Suppose $\mathbf{x}$ solves the least squares problem, i.e., $\|A\mathbf{x} - \mathbf{b}\|_2^2$ is minimized while $\mathbf{x} + \alpha\mathbf{z}$ is not a solution to the least squares problem. Then, using Part (a), deduce $A^\top(A\mathbf{x} - \mathbf{b}) = \mathbf{0}$, i.e., $\mathbf{x}$ is a solution to the normal equation $A^\top A\mathbf{x} = A^\top\mathbf{b}$.

(c) Suppose both $\mathbf{x}$ and $\mathbf{x} + \alpha\mathbf{z}$ are both solutions to the same least squares problem. Show that $\mathbf{z} \in \mathit{null}(A)$.

(d) If A is full rank, then show that such $\mathbf{z}$ must be $\mathbf{0}$. This concludes the proof of the statement we wanted to prove, i.e., the normal equation has a unique solution, i.e., $A^\top A$ is nonsingular.

Exercise 3

This exercise is at a similar level to possible midterm exam problems, and is worded as such as well. However, in this problem you need to use a computer / calculator to report your findings. On the exam, you will be not have a computer or calculator, and will be asked to report some answers as expressions and equations.

Suppose we are running an experiment to determine the energy potential $E(y)$ of particular type of particle traveling along a vertical line. We know that the energy E should be a quadratic function in the position y, i.e.

$$E(y) = ay^2 + by + c.$$

We measure energy of the particle at six different positions and obtain the following data:

y	2	4	6	8	10	12
E	4.0	0.4	0.6	3.8	10.1	17.5

(a) Write down the (overdetermined) system given by these data. (Please use the variables A, x,and b as we did in class.)

(b) Write down the normal equations for this system explicitly, i.e. as a system of linear equations.

(c) Use MATLAB to compute the (unique) solution $\hat{x}$ to your normal equations. Based on your results, what is the best-fit quadratic for the energy potential $E(x)$?

(d) Compute the condition number of this system explicitly. That is, you can use MATLAB to help you do calculations (e.g. of inverse, matrix multiplications, etc.), but you should not use the `cond` command except possibly to check your work. You must show your work.

(e) Based on your model, at what position y_* should the energy of the particle be lowest?