MAT167 HW04

Due 5/4/23 at 11:59 pm on Gradescope

Instructions You may handwrite or type up your homework solution. Regardless, your solutions **must be neat**. If your solutions are incoherent, illegible, or difficult to read, you will lose style points during grading.

- You must justify your answers, i.e. show sufficient steps so that the grader can see that you understand the algorithm. Aside from calculations, your solutions should be written in **full sentences.**
- Create PDFs for your solutions. If you have handwritten work, you need to scan these (e.g. using a scanner app); you should export any code or computer output to a PDF as well. Finally, combine these PDFs into **one single PDF**.
- Submit the **one single PDF** to Gradescope. If you decide to change your solution, you can update your PDF submission as many times as you'd like, as long as the deadline hasn't passed.

Covered material This homework covers our review of projectors as well as §4.1-4.2 of the book.

Exercise 1

Prove that if P is an *orthogonal* projector, then $I - 2P$ is an orthogonal matrix.

You should use the following equivalent definition of orthogonal matrices:

Theorem. A square matrix Q is *orthogonal* if and only if $Q^T = Q^{-1}$, i.e. $Q^\top Q = I$.

Exercise 2

The $x, y \in \mathbb{R}^m$ where $\|x\|_2 = \|y\|_2$. Recall that a Householder transformation is a transformation of the form

$$H = I - \frac{2vv^\top}{\|v\|_2^2} = I - 2uu^\top \qquad \text{where } u = \frac{v}{\|v\|}.$$

I've changed the notation for the matrix to H for "Householder". Your book uses P. We will use P to reference an actual projector in this problem.

In class, we showed how to obtain a Householder transformation (a reflection matrix) by projecting x onto the line through both x and y (i.e. in the direction of $v = x - y$), and then subtracting that projection from x twice.

In this exercise, you will verify the geometric explanation above and compute an example in $\mathbb{R}^3$.

(a) Prove that if u is a unit-length vector, then the outer product $P = uu^\top$ is an *orthogonal projector*. (Therefore, by Exercise 1, R is an orthogonal matrix.)

(b) Let

$$x = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \qquad \text{and} \qquad y = \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}.$$

Convince yourself that x and y are the have the same length. Compute the Householder matrix H such that $Hx = y$.

(c) What is the Householder vector u in this case?

(d) The matrix H represents a reflection across a plane in $\mathbb{R}^3$. This plane is the nullspace of the orthogonal projection P (which projected onto the line spanned by v). Describe a basis for this *plane of reflection*. (Hint: Use the complementary projector!)