MAT167 HW05

Due 5/11/23 at 11:59 pm on Gradescope

Instructions You may handwrite or type up your homework solution. Regardless, your solutions **must be neat**. If your solutions are incoherent, illegible, or difficult to read, you will lose style points during grading.

- You must justify your answers, i.e. show sufficient steps so that the grader can see that you understand the algorithm. Aside from calculations, your solutions should be written in **full sentences.**
- Create PDFs for your solutions. If you have handwritten work, you need to scan these (e.g. using a scanner app); you should export any code or computer output to a PDF as well. Finally, combine these PDFs into **one single PDF**.
- Submit the **one single PDF** to Gradescope. If you decide to change your solution, you can update your PDF submission as many times as you'd like, as long as the deadline hasn't passed.

Covered material This homework covers orthogonal projectors, Gaussian Elimination with partial pivoting, and QR decomposition (lecture notes, §4.1, §5.1).

Exercise 1

Consider the nonsingular matrix $A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}$. Compute the following by hand:

(a) The orthogonal projector P_A onto range(A)

(b) The image under P_A of the vector $v = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$

Exercise 2

Let $A \in \mathbb{R}^{m \times n}$, where $m \geq n$, and suppose its QR decomposition is

$$A = QR.$$

Show that $\|A\|_F = \|R\|_F$. (Note that it doesn't matter if you think of R as the matrix from the full QR decomposition or the thin one.)

Exercise 3

Consider the "flipping" operation on vectors in $\mathbb{R}^m$, which reverses the coordinates of a vector $x \in \mathbb{R}^m$:

$$F : \mathbb{R}^m \to \mathbb{R}^m$$
$$\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{bmatrix} \mapsto \begin{bmatrix} x_m \\ x_{m-1} \\ \vdots \\ x_1 \end{bmatrix}$$

(a) Note that F is a permutation of the coordinates of x, and hence is really a permutation matrix $F \in \mathbb{R}^{m \times m}$. Write down the flipping matrix F on $\mathbb{R}^4$; then write down the form of the flipping matrix F on $\mathbb{R}^m$ for arbitrary m. (This should be very explicit, i.e. you should know every entry.)

(b) Now define a new linear transformation

$$E = \frac{1}{2}(I + F) \qquad \text{i.e.} \qquad Ex = \frac{x + Fx}{2}.$$

Is E and orthogonal projector, an *oblique projector* (i.e. a projector that is not an orthogonal projector), or not a projector at all? What are the entries of $E \in \mathbb{R}^{m \times m}$?

(c) The vectors in range(E) have a very special property; what is it? Explain how to came to your answer. *Hint: E stands for "even", in the sense that a function like* $\cos(x)$ *is even, because it is symmetric.*

(d) The vectors in null(E) also satisfy a very special property; what is it, and how do you know? *Hint: The opposite of "even" is "odd", in the sense that a function like* $\sin(x)$ *is odd, because it is "antisymmetric".*