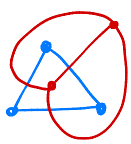
Lecture 7

Announcements: I'm giving a 2-part lecture series @ MathClub + AWM this week. M 5:30 – 6:30 pm in MSB Colloquium Room (MSB 1147)

defn. The planar dual of a planar graph Γ is given by

- V = regions of the plane cut out by Γ
- E = edge b/w v_1 and v_2 for every edge in Γ that is part of the border between them

eg.  the blue and red planar graphs are dual to each other

Q. Let D be a knot diagram. View D as a graph where

- vertices = crossings
- edges = arcs of the knot diagram

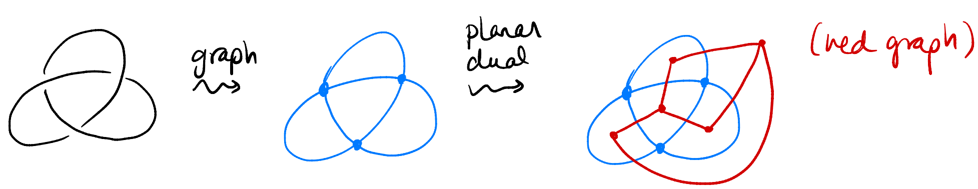
What do you notice about the planar dual of D?

A. Since each vertex of D has valence 4, the dual graph

(valence: # edges coming out of it)

is composed of 4-cycles (ie squares □),

ie the boundary of each region of the dual graph is a 4-cycle.

eg. 

Sketch of proof of checkerboard coloring existence

- dual graph made of squares
- color vertices along spanning tree
- verify that adding edges doesn't cause obstruction

Prop. Every knot diagram D has a checkerboard coloring.

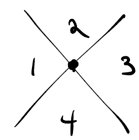
Proof sketch

Let D be a knot diagram; let Γ be the dual graph to D.

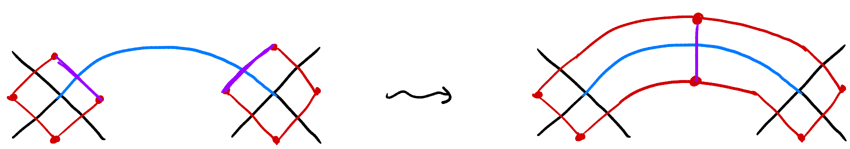
Claim A Γ is composed of 4 cycles glued together along edges.

Pf of Claim

- each vertex (crossing) of D is 4-valent.
- the four regions touching at a crossing must all be distinct



- there is a 4-cycle in Γ surrounding each vertex
- these cycles are glued along their edges, wherever D had an edge connecting two crossings



because of the blue edge in D, the purple arcs in the left picture are actually the same edge in the dual graph Γ.

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It suffices to color the vertices of Γ 2 colors, so that if $(v_1, v_2) \in E(\Gamma)$, then v_1 and v_2 are colored differently.

Algorithm: Pick a vertex and color a random color. Then propagate. There is only a constraint if you travel along a loop (ie cycle) to a vertex you've already colored.

But by the claim, every cycle in Γ is even length (prove this). So your coloring will be consistent. ∎

Knot Polynomials via skein relations

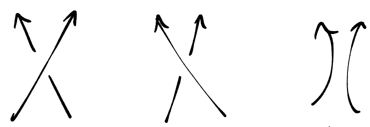
Alexander Polynomial (1923)

$\Delta_K(t) \in \mathbb{Z}[t, t^{-1}]$ Laurent polynomials, K oriented

Defined recursively:

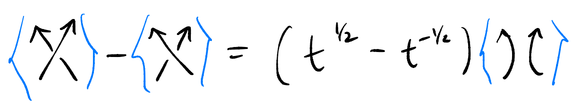
- $\Delta_U(t) = 1$
- $\Delta_{L_+}(t) - \Delta_{L_-}(t) = (t^{1/2} - t^{-1/2})\,\Delta_{L_0}(t)$

i.e.



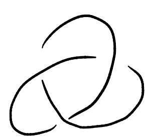
L_+ L_- L_0

local pictures



shorthand.

eg. Resolving tree:



Fact $\Delta_K(t^{-1}) = \Delta_K(t)$ for all knots (symmetric / palindromic)

Pf. look at skein relation...