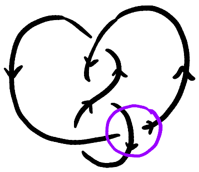
LECTURE 8 Knot Polynomials via skein relations (~ Chp. 6)

Q. 

Draw the diagrams resulting from replacing the positive crossing (in purple circle) with

① a negative crossing

② an oriented smoothing ie (NOT)

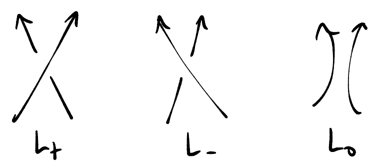
Alexander Polynomial (1923)

$\Delta_K(t) \in \mathbb{Z}[t, t^{-1}]$ Laurent polynomials, K oriented

Defined recursively:

- $\Delta_U(t) = 1$
- $\Delta_{L_+}(t) - \Delta_{L_-}(t) = (t^{1/2} - t^{-1/2})\,\Delta_{L_0}(t)$

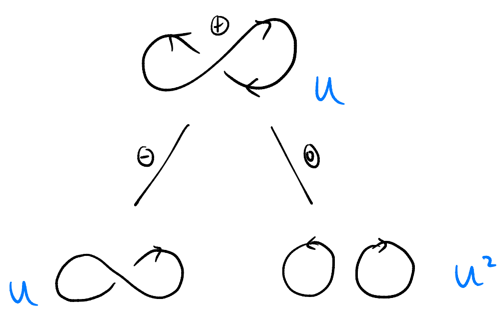
i.e.



local pictures

$\{\ \} - \{\ \} = (t^{1/2} - t^{-1/2})\{\ \}$ shorthand

eg. Resolving tree:



$\Rightarrow \quad U - U = (t^{1/2} - t^{-1/2})\,U^2 \quad \Rightarrow \quad \Delta_{U \sqcup U}(t) = 0$

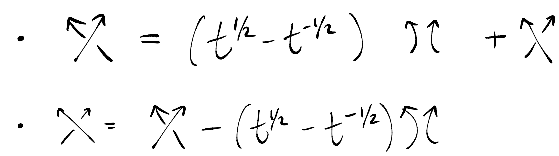
Fact $\Delta_K(t^{-1}) = \Delta_K(t)$ for all knots (symmetric / palindromic)

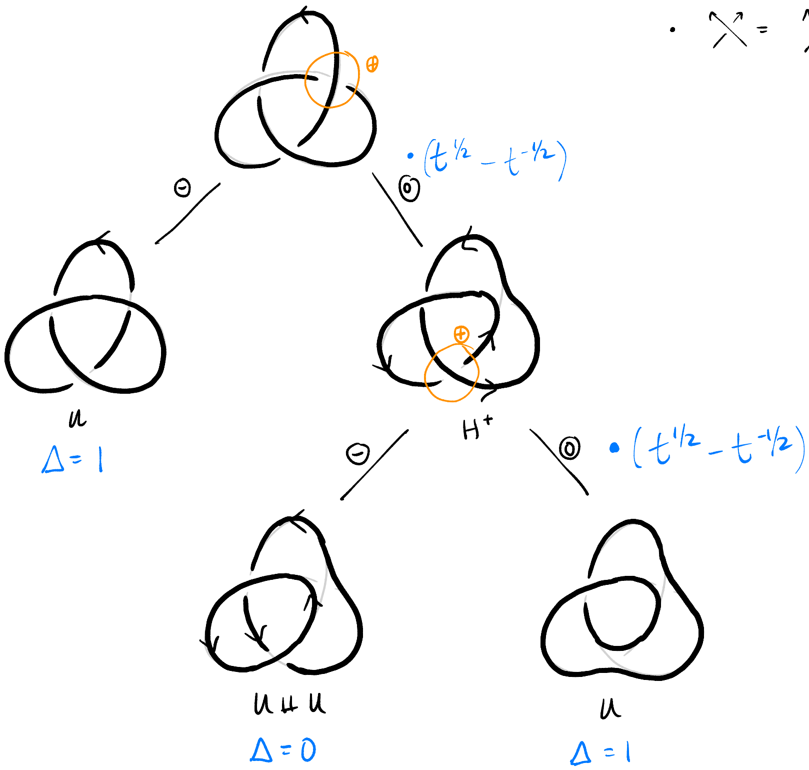
Pf. look at skein relation...

(keep on board)

eg. <u>Resolving tree:</u>

<u>Useful:</u>





$$\Rightarrow \Delta_T(t) = 1 + (t^{1/2} - t^{-1/2})\left[0 + (t^{1/2} - t^{-1/2})\right] = 1 + (t - 2 + t^{-1}) = t - 1 + t^{-1}$$

Jones Polynomial (1984)

- $V(\bigcirc) = 1$

$$q^{-2}\,\infty \;-\; q^{2}\,\infty = (q-q^{-1})\,\circ\circ \;\Rightarrow\; \circ\circ = \frac{q^{-2}-q^{2}}{q-q^{-1}} = -(q+q^{-1})$$

- $t^{-1}V_{L_+} - t\,V_{L_-} = (t^{1/2}-t^{-1/2})\,V_{L_0}$

$q^{-2}V_{L_+} - q^{2}V_{L_-} = (q-q^{-1})\,V_{L_0}$

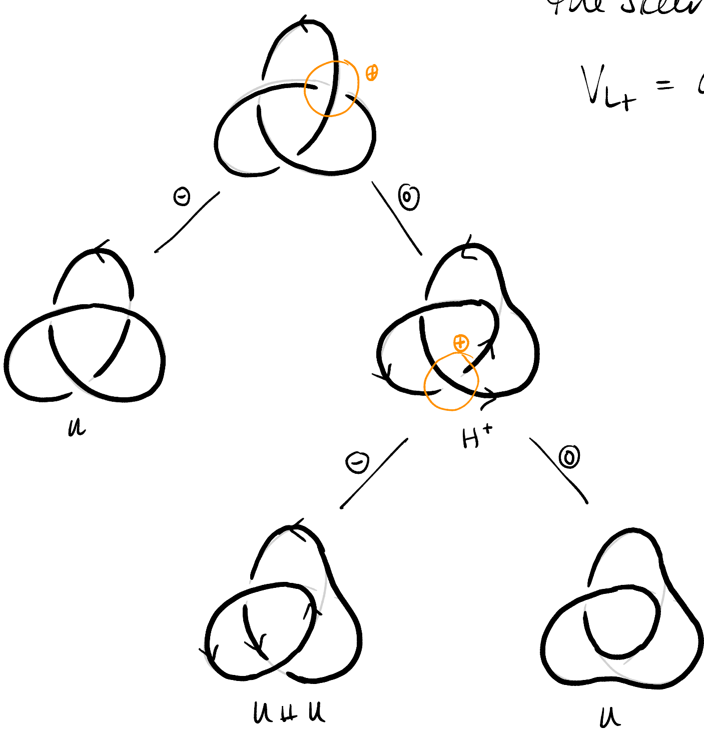
let $q = t^{1/2}$

eg. Right-handed trefoil

Resolving tree:

Since we have 3 ⊕ crossings, let's rewrite the skein relation as

$$V_{L_+} = q^{2}\left[q^{2}V_{L_-} + (q-q^{-1})V_{L_0}\right]$$



$$V_T = q^{2}\left[q^{2}V_u + (q-q^{-1})V_{H^+}\right] = q^{4} + (q^{3}-q)V_{H^+}$$

$$V_{H^+} = q^{2}\left[q^{2}V_{oo} + (q-q^{-1})V_u\right] = q^{2}\left[q^{2}(-(q+q^{-1})) + (q-q^{-1})\right]$$

$$= q^{2}\left[-q^{3} - \cancel{q} + \cancel{q} - q^{-1}\right] = q^{2}(-q^{3}-q^{-1}) = -q^{5} - q$$

$$\Rightarrow V_T = q^{4} + (q^{3}-q)(-q^{5}-q) = \cancel{q^{4}} - q^{8} - \cancel{q^{4}} + q^{6} + q^{2} = -q^{8} + q^{6} + q^{2}$$

i.e. $V_T(q) = -q^{8}+q^{6}+q^{2}$, $V_T(t) = -t^{4}+t^{3}+t^{2}$

HOMFLY-PT (1985/1987)

- $P_u = 1$
- $\alpha P_{L_+}(\alpha, z) - \alpha^{-1} P_{L_-}(\alpha, z) = z P_{L_0}(\alpha, z)$

Relation to Alexander + Jones:

- $\Delta(z) = P(\alpha = 1, z = t^{1/2} - t^{-1/2} = q - q^{-1})$
- $V(t) = P(\alpha = t^{-1}, z = t^{1/2} - t^{-1/2})$