LECTURE 12

No review problem! Everyone gets 1/1 Participation today.

<u>Recall</u> If we build up a surface F using

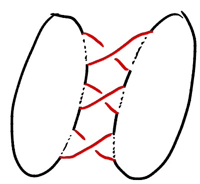
v vertices, e edges, and f faces,

then the Euler characteristic $\chi(F) = v - e + f$.

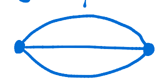
If Σ is a closed, orientable surface, then $\chi(\Sigma) = 2-2g$.

<u>Claim</u> If you build a surface F out of v disks and e bands

then $\chi(F) = v - e$.

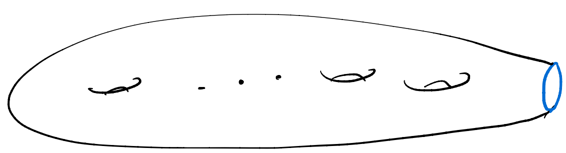


* We didn't prove this at all. This has to do with the fact that this surface is "homotopy equivalent" to



If F = the Seifert surface for the trefoil above, then

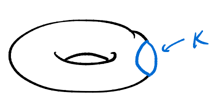
$\chi(F) = 2-3 = -1$. Let $\hat{F} = F \cup D^2$, where D^2 is a disk glued on to the boundary of this abstract surface



K is abstractly just a circle

Then $\chi(\hat{F}) = \chi(F) + 1 = 2-3+1 = 0 = 2-2g$, so $g=1$.

So F is homeomorphic to a torus with one boundary component:

$F \cong$ 

In general, if F is a Seifert surface for K, then

$\chi(\hat{F}) = \chi(F) + 1 = 2 - 2g(F)$ $\rightsquigarrow$ we can solve for $g(F)$.

Goal Define a knot invariant

$$g_3(K) = \underbrace{3\text{-genus}}_{} \text{ of } K = \min \left\{ g(F) \;\middle|\; \begin{array}{l} F \text{ is a Seifert surface} \\ \text{for } K \end{array} \right\}$$

b/c our surfaces are embedded in $\mathbb{R}^3$

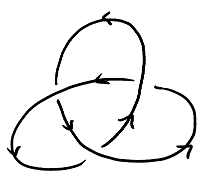
But first... how do we know Seifert surfaces always exist?

Seifert's Algorithm

Input: $D(K)$, a diagram of a knot K.

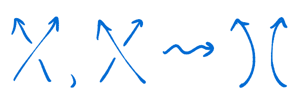
Output: an orientable surface in $\mathbb{R}^3$ made up of disks and bands

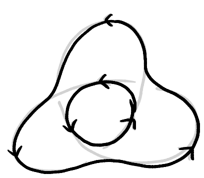
eg. Consider the usual diagram of the right-handed trefoil: and pick an orientation.



Draw the oriented smoothing of D, a complete resolution of D obtained by choosing the oriented smoothing of each crossing

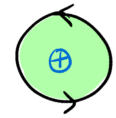
ie.



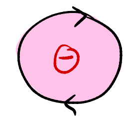


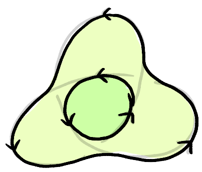
Now think of each circle as the boundary of a disk.
If circles are nested the inner one is viewed as "above" the outer one (ie closer to you).

- If a disk's ∂ runs CCW, then you are looking at the "front" (or "positively oriented") side of it



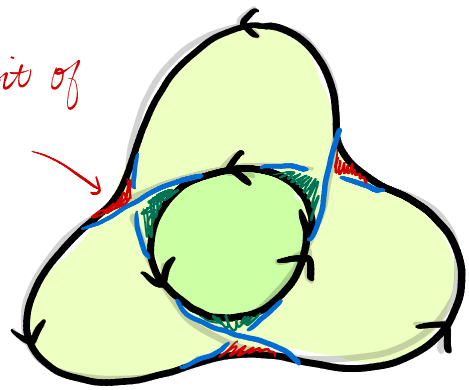
- If a disk's ∂ runs CW, then —— "back", ie "negatively oriented"





Now attach bands to the disks so that the boundary of the resulting surface looks like the diagram D:

Only see a little bit of the "back" side



ie add the crossings back into the picture

Note # bands = # crossings!

thm The resulting surface will always be orientable.

(we'll think about why later.)

Q. What would have happened if I chose the other orientation at the beginning?

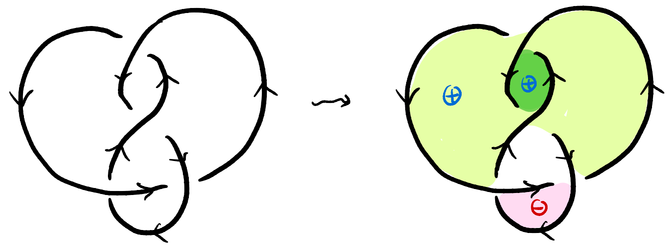
A. front and back sides switch (doesn't change the surface).

Q. What is the genus of the surface we made?

A. $2-2g = 2$ disks $- 3$ bands $+ 1$ ∂ component $= 0$

$\Rightarrow g(F) = 1$. (This turns out to be $= g(\text{trefoil})$!)

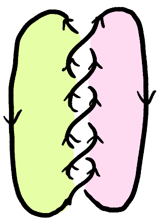
Another example



Note how I can just shade the diagram to see the surface

$$2-2g = 3-4+1 = 0$$

$g(F)=1 \quad (= g(\text{Fig}8)!)$



5 crossing instead of 3

$2-2g = 2$ disks -5 crossings $+ 1$ ∂ cpnt $= -2$

$\Rightarrow g=2$

If there is time...

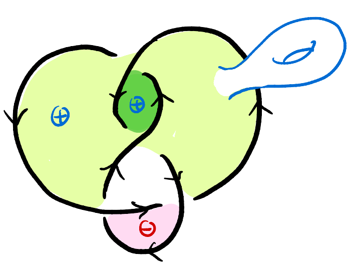
prop. Every knot has infinitely many Seifert surfaces.

pf.

Let F be a Seifert surface for K, and let $g = g(F)$.

Then $F \# T^2$ is a genus $g+1$ Seifert surface for K:

eg.



← still orientable!

In fact, $\underbrace{F \# T^2 \# \cdots \# T^2}_{k \text{ copies}} = F \#^k T^2$ has genus $g+k$.

Since they all have different χ, they are all non-homeomorphic! ☒