LECTURE 14

No participation slips until further notice. (We're not an evening class, but some students commute and it makes no sense to come for fewer classes.)

Stay safe!

Last time Given surface $F \rightsquigarrow$ "linking matrix"

In particular, if F is a Seifert surface for K, this is the "Seifert matrix" or "Seifert form". (like an intersection matrix, but using linking)

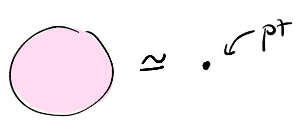
Goal: What are the curves γ_i we will use to compute the Seifert matrix?

The rows and columns of the Seifert matrix are a basis for $H_1(F)$, the first homology group of F.

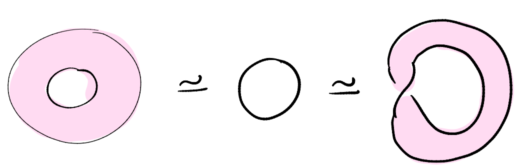
Aside What are homology groups? – think vector space instead of groups

- Given top space X, the homology of X is a seqn of vector spaces $H_i(X)$ for $i \in \{0, 1, \ldots\}$
- This is a homotopy invariant ie even coarser than "homeomorphism invariant":

rough defn Two spaces X, X' are homotopy equivalent if you can deform one into the other; you are even allowed to squish down the dimension!

eg. 

Since we can squish $\mathbb{D}^2$ down to a pt, $\mathbb{D}^2 \simeq *$. (I don't need to show you how to get back!)

eg. 

Annulus $\simeq$ circle $\simeq$ Mobius band

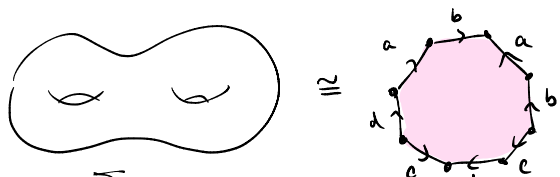
(but these are all $\not\cong$, ie non homeomorphic)

Fact χ (Euler char) is also a homotopy invariant!

In fact, $\chi(X) = \sum_{i=0}^{\infty} (-1)^i \dim H_i(X)$

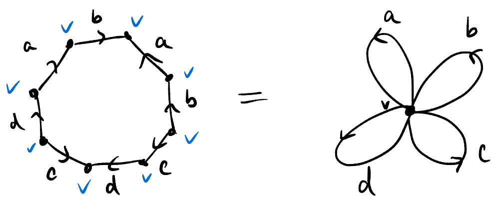
To understand $H_1(X)$, we just need to look at its "1-dim" skeleton:

eg.



Σ

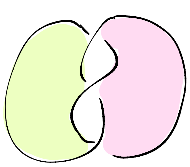
1-skeleton of Σ =



Since we see 4 "loops", we say the VS $H_1(X) = \text{span}\{a, b, c, d\}$

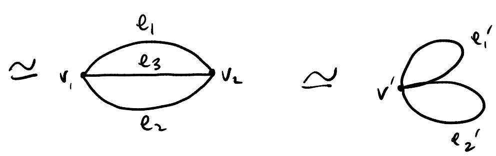
* It's a VS: what loop is $a+b$? $a-b$?

eg. F

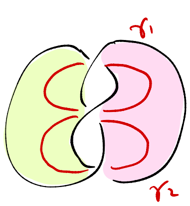


This Seifert surface for 3_1 is already homotopy equivalent to its own 1-skeleton!

Basis for loops in this graph:



Now find loops in F that represent e_i' (ie were shrunken down to the e_i').



Note. we don't need the representatives γ_i to meet at a point or anything; in fact, it's better if they don't.

(ie intersect only transversely, so that at each intersection the intersection # is ± 1)

Then $H_1(F) = \text{span}\{\gamma_1, \gamma_2\}$!

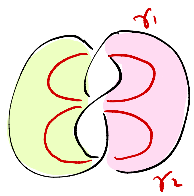
(Actually, choose some orientations too; ie choose either γ_1 or $-\gamma_1$ in the VS as a basis vector)

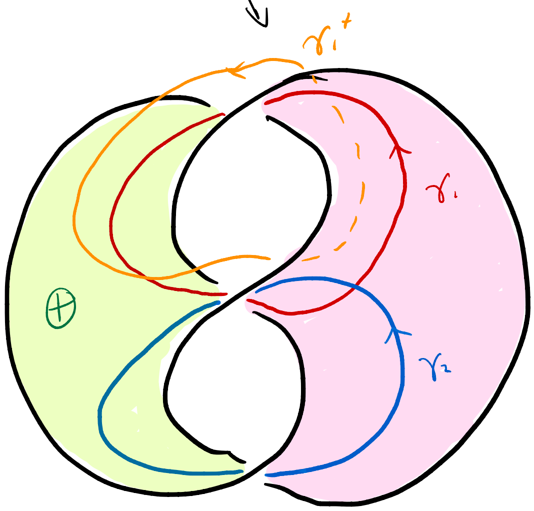
Full Example

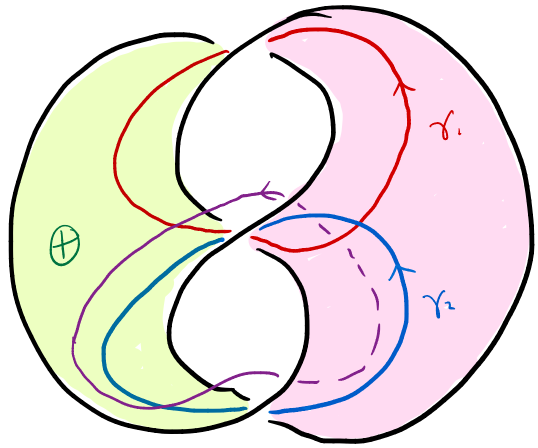
Seifert matrix:

$$\begin{array}{c|cc} & \gamma_1^+ & \gamma_2^+ \\ \hline \gamma_1 & lk(\gamma_1,\gamma_1^+) & lk(\gamma_1,\gamma_2^+) \\ \gamma_2 & lk(\gamma_2,\gamma_1^+) & lk(\gamma_2,\gamma_2^+) \end{array} = \begin{array}{c|cc} & \gamma_1^+ & \gamma_2^+ \\ \hline \gamma_1 & & \\ \gamma_2 & & \end{array}$$

F







$lk(\gamma_1,\gamma_1^+) = -1$

$lk(\gamma_2,\gamma_1^+) = 0$

$lk(\gamma_1,\gamma_2^+) = +1$

$lk(\gamma_2,\gamma_2^+) = -1$

$\Rightarrow \quad V = \begin{bmatrix} -1 & 1 \\ 0 & -1 \end{bmatrix}$

Unfortunately this is the standard notation

<u>Cool trick:</u> What is $\det(V - tV^T)$?

$$V - tV^T = \begin{bmatrix} -1 & 1 \\ 0 & -1 \end{bmatrix} - t\begin{bmatrix} -1 & 0 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} -1+t & 0 \\ -t & -1+t \end{bmatrix}$$

$$\det(V - tV^T) = (-1+t)^2 + t = (t-1)^2 + t = (t^2 - 2t + 1) + t = t^2 - t + 1$$

$\rightsquigarrow$ shift by overall power of t until symmetric (palindromic)

$\Rightarrow \Delta_K(t) = t - 1 + t^{-1}$!