Lecture 15

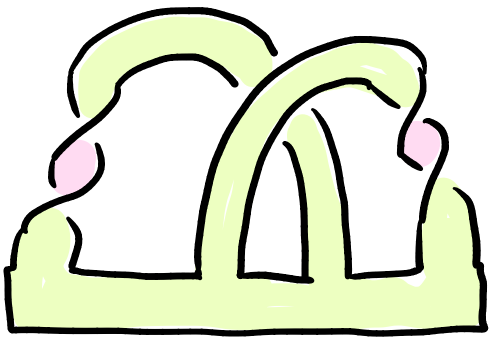
Return to usual participation policy starting next Monday, 5/8. (Week 6)

It's all linear algebra! Continue with surfaces - familiarity.

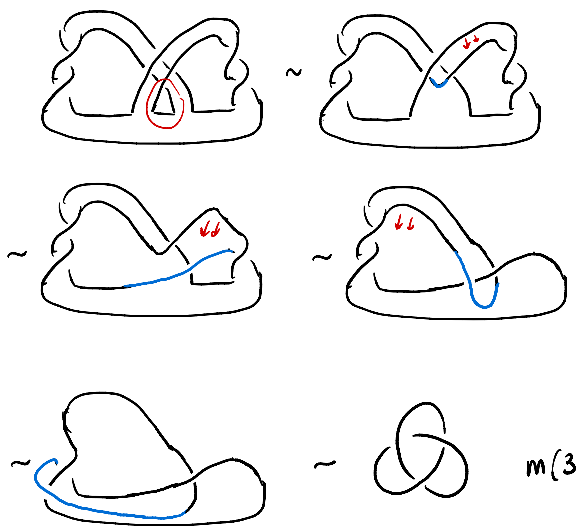
I. Viewing Seifert surfaces as manifolds built up from handles (orientable)

Q. How do our choices affect V?

F



$\partial F =$

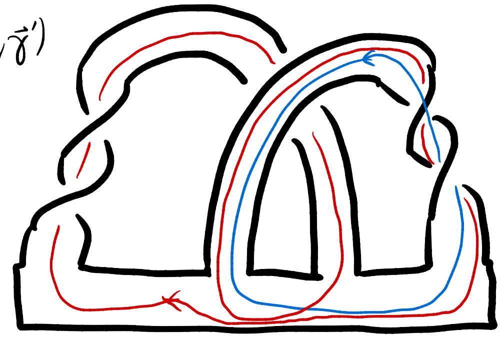


$m(3_1)$

- We chose a basis for $H_1(F)$. We need some equivalence to account for this!

$(F, \vec{\gamma})$

$(F, \vec{\gamma}')$



$$V = \begin{array}{c c} & \begin{array}{cc} \gamma_1^+ & \gamma_2^+ \end{array} \\ \begin{array}{c} \gamma_1 \\ \gamma_2 \end{array} & \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \end{array}$$

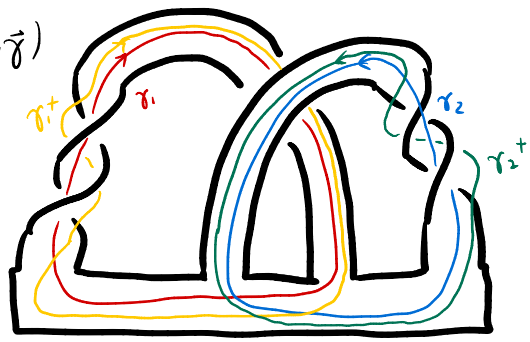
$$V' = \begin{array}{c c} & \begin{array}{cc} \gamma_1'^+ & \gamma_2'^+ \end{array} \\ \begin{array}{c} \gamma_1' \\ \gamma_2' \end{array} & \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} \end{array}$$

Key: Changing your basis $\{\gamma_1, \gamma_2\} \longrightarrow \{\gamma_1', \gamma_2'\}$ changes the Seifert matrix by $V' = P^T V P$ where P is invertible.

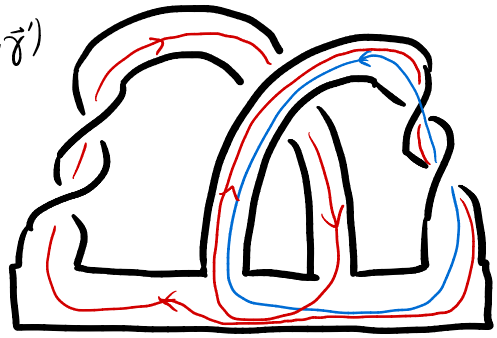
e.g. $$V' = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}}_{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}} \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$$

We can see this change of basis in the picture:

$(F, \vec{\gamma})$



$(F, \vec{\gamma}')$



$\gamma_1' = \gamma_1 - \gamma_2$

$\gamma_2' = \gamma_2$

$$\begin{tikzcd} \gamma_1 \arrow[r, "1"] \arrow[dr, "-1"] & \gamma_1 \\ \gamma_2 \arrow[r, "1"'] & \gamma_2 \end{tikzcd}$$

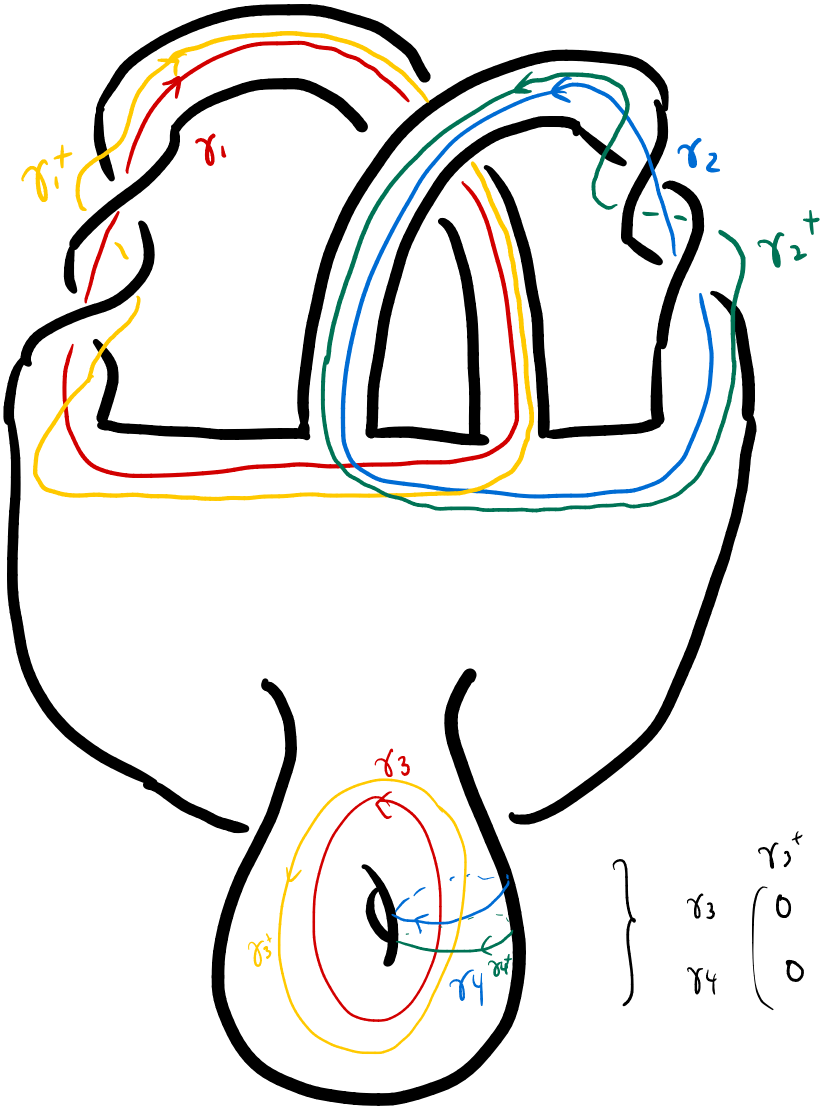
$$P^T = \begin{array}{c c} & \begin{array}{cc} \gamma_1 & \gamma_2 \end{array} \\ \begin{array}{c} \gamma_1 \\ \gamma_2 \end{array} & \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} \end{array}$$

$\Rightarrow$ change of basis matrix is "Similar matrices"

$$\begin{tikzcd} \langle \gamma_1', \gamma_2' \rangle \arrow[r, "P"] \arrow[rrr, bend right, "V'"'] & \langle \gamma_1, \gamma_2 \rangle \arrow[r, "V"] & \langle \gamma_1, \gamma_2 \rangle \arrow[r, "P^T"] & \langle \gamma_1', \gamma_2' \rangle \end{tikzcd}$$

we could have also chosen a diff. Seifert surface all together.

eg. Stabilization / Destabilization



$$\begin{array}{cc} & \begin{matrix}\gamma_3^+ & \gamma_4^+\end{matrix} \\ \begin{matrix}\gamma_3 \\ \gamma_4\end{matrix} & \begin{pmatrix}0 & 1 \\ 0 & 0\end{pmatrix}\end{array}$$

Choose γ_3, γ_4 so that there is the sub-linking matrix.

New Seifert Matrix

$$\begin{array}{c|cc|cc} & \gamma_1 & \gamma_2 & \gamma_3 & \gamma_4 \\ \hline \gamma_1 & & & & \\ & \multicolumn{2}{c|}{V} & \multicolumn{2}{c}{*} \\ \gamma_2 & & & & \\ \hline \gamma_3 & \multicolumn{2}{c|}{\multirow{2}{*}{$*$}} & 0 & 1 \\ \gamma_4 & & & 0 & 0 \end{array}$$

I could have the new torus # summand wind around existing bands like crazy

HW06: What happens to $\det(V - tV^T)$?

Fact

① Any two Seifert surfaces for K are related by stabilization/destabilization.

② Any 2 Seifert matrices from the same Seifert surface are similar (in lin alg sense).

thm (Seifert) Any two Seifert matrices for K are related by a sequence of the above moves.

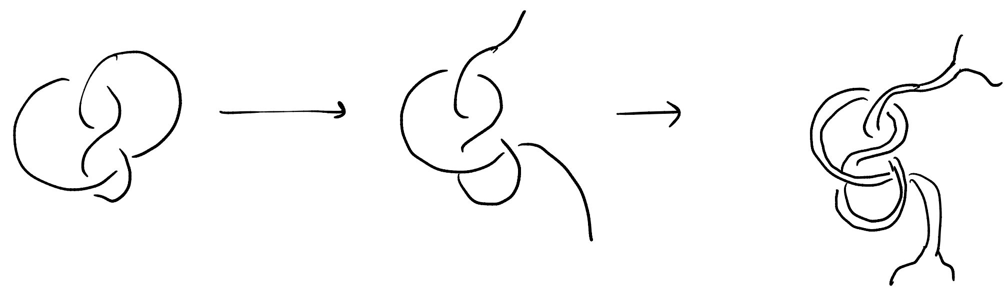
$\rightsquigarrow$ defines "S-equivalence" on matrices

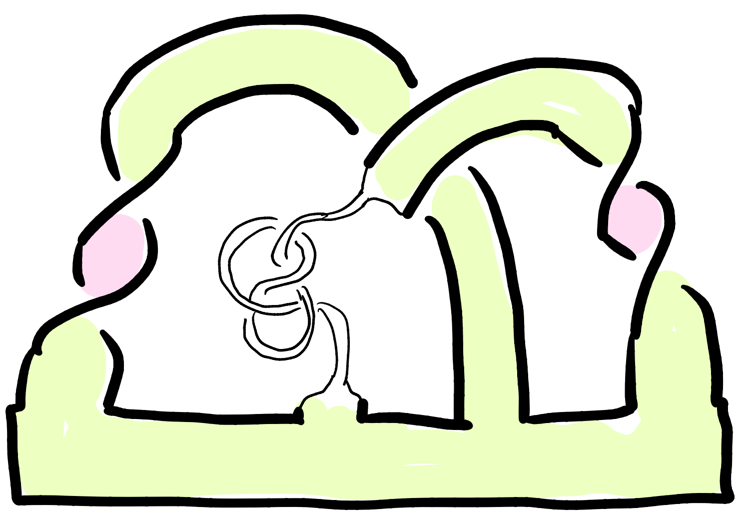
Q. Can any matrix be a Seifert matrix? (I didn't think about this)

Note S-equiv class is NOT a complete knot invariant.

proof tie a writhe 0 ribbon into one of the bands.

eg.





$lk(\tilde{\gamma}_2, \tilde{\gamma}_2^{+}) = lk(\gamma_2, \gamma_2^{+})$

($\tilde{\gamma}_2$ is the same as γ_2 except follows the tangled ribbon)

Clearly the tangle doesn't link with the other bands.