LECTURE 16

Surfaces in 4D; sliceness
decomposing surfaces

Q. How are you doing? Any concerns about how the class is going?

* I had planned on grid diagrams but let's do more topology and move toward the kind of math and calculations I do!

Ⓘ Which category are we in?

Make more precise what objects and relationships we are working with.

<u>Categories</u>

<u>defn.</u> A category $\mathcal{C}$ is the data

- $Ob(\mathcal{C})$ = objects of $\mathcal{C}$ (a collection)
- $Mor(\mathcal{C})$ = morphisms between objects in $\mathcal{C}$:

  Given $X, Y \in Ob(\mathcal{C})$,

  $Mor_{\mathcal{C}}(X,Y)$ is a set of morphisms $A: X \to Y$.

  that behave like functions:

  - $Id_X \in Mor_{\mathcal{C}}(X,X)$
  - composition operation:

    If $A \in Mor_{\mathcal{C}}(X,Y)$, $B \in Mor_{\mathcal{C}}(Y,Z)$,

    there is a morphism $B \circ A \in Mor_{\mathcal{C}}(X,Z)$

    such that the following <u>diagram commutes</u>:

$$\begin{tikzcd} X \arrow[r, "A"] \arrow[rr, bend left, "B \circ A"] & Y \arrow[r, "B"] & Z \end{tikzcd}$$

  - also composition is associative

Familiar example: vector spaces over $\mathbb{C}$ (or $\mathbb{R}$, if you'd like)

$\mathbb{C}$-Vect or $\text{Vect}_{\mathbb{C}}$ = the category of vector spaces over $\mathbb{C}$

- $\text{Ob}(\text{Vect}_{\mathbb{C}}) = \{$ all vector spaces over $\mathbb{C}$, of any dimension $\}$

  there are tons of vector spaces for each dimension!

- $\text{Mor}(\text{Vect}_{\mathbb{C}}) = \{$ linear transformations $\}$

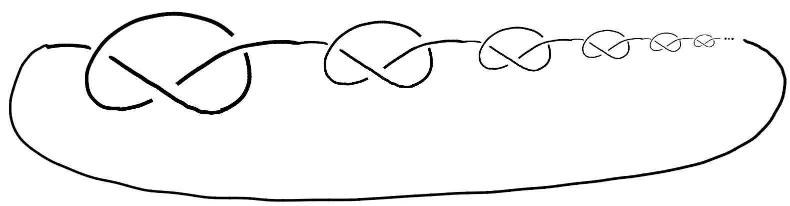
  $\text{Mor}(V, W) = \{$ linear transformations $A : V \longrightarrow W \}$

  Composition: $\underset{\text{"B"}}{\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}} \underset{\text{"A"}}{\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}} = \underset{\text{"BA"}}{\begin{bmatrix} 2 & 1 \\ 4 & 2 \end{bmatrix}}$

Back to Knots

We have been working with smooth knots and links.

eg. Wild knots are not allowed!

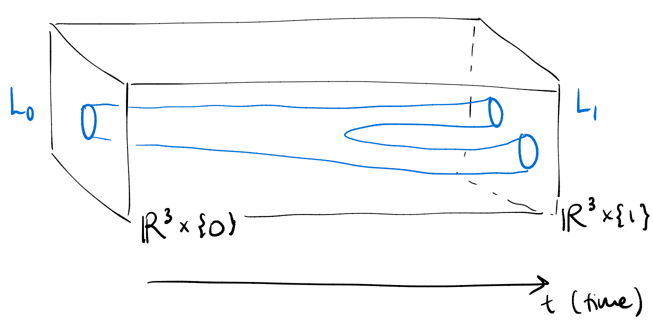


Our category: $\mathcal{C}$ = (Links, link cobordisms) * not up to isotopy!

$\text{Ob}(\mathcal{C})$ = smooth links $L : S^1 \hookrightarrow \mathbb{R}^3$ (infinitely differentiable)

$\text{Mor}_{\mathcal{C}}(L_0, L_1) = \{$ cobordisms $F : L_0 \longrightarrow L_1 \}$ (up to ambient isotopy)

defn. A (smooth) cobordism F from L_0 to L_1 is a (smooth) surface embedded in $\mathbb{R}^3 \times [0,1]$ such that $\partial F = L_0 \cup L_1$, where $L_i \subset \mathbb{R}^3 \times \{i\}$



But! We've actually been drawing diagrams!

<u>Our actual category:</u> $\mathcal{C} = \text{Diag}$

- $Ob(\mathcal{C}) = \{$ diagrams of links in $\mathbb{R}^2 \}$ * not up to isotopy!
- For $D_0, D_1 \in Ob(\mathcal{C})$,

  $Mor_{\mathcal{C}}(D_0, D_1) = \{$ movies relating D_0 to D_1 that represent link cobordisms $\}$ up to "movie moves" of Carter-Saito

↪ I haven't told you enough about the morphisms yet to understand this category

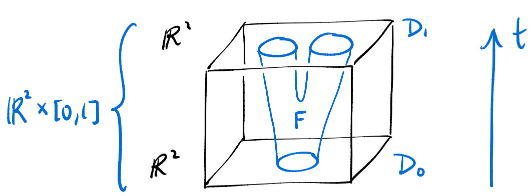
Let's start with a simpler, but related category:

eg. $\mathcal{C}$ = "planar circles"

$Ob(\mathcal{C})$ = smoothly embedded finite collections of surfaces in $\mathbb{R}^2$:

$$\coprod_{i=1}^{n} S^1 \xhookrightarrow{\text{smooth!}} \mathbb{R}^2$$

example morphism:

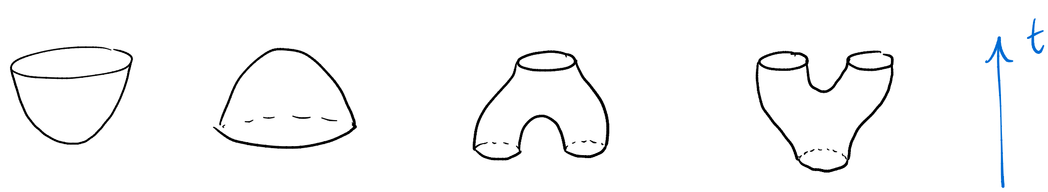


The cobordism (smooth!)

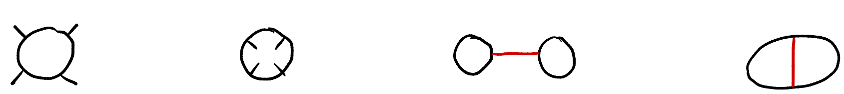
$(F : D_0 \longrightarrow D_1) \in Mor_{\mathcal{C}}(D_0, D_1)$.

In this category, the morphisms can all be decomposed into elementary pieces:

cobordism



diagrammatic shorthand



name   birth   death   merge   split