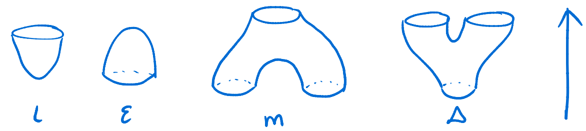
# LECTURE 17

I'm pushing you all — its okay if it takes time!

Q. Build the torus  out of elementary cobordisms

birth, death, merge, split 


In two different ways.

Summary of last time (a lot to take in)

- **Categories** $Ob(\mathscr{C})$ = collection of objects

  For $X, Y \in Ob(\mathscr{C})$,

  $Mor_{\mathscr{C}}(X,Y)$ = (set) collection of structure-preserving morphisms $X \longrightarrow Y$

  such that
  - $Id_X \in Mor_{\mathscr{C}}(X,X)$
  - $\circ : Mor(X,Y) \times Mor(Y,Z) \longrightarrow Mor(X,Z)$
    $(f,g) \longmapsto g \circ f$
  - $\circ$ is associative

eg. In MAT108, you learned about sets. Sets = (sets, functions b/w sets)

Q. How large is the morphism set $Mor_{Sets}([1],[2])$?

What about $Mor([2],[2])$, $Mor([3],[5])$?

- **Cobordisms** (I am only working with smooth things!)

  example standard defn:

  A cobordism from an n-dim'l manifold $X^n$ to another $Y^n$ is an $(n+1)$-dim'l manifold $W^{n+1}$ such that $\partial W^{n+1} = -X^n \amalg Y^n$.

- **Embedded** 2-dim'l **cobordism** b/w 1-manifolds embedded in $\mathbb{R}^3$:

  $F \subset \mathbb{R}^3 \times [0,1]$ is a cobordism from $L_0 \subset \mathbb{R}^3 \times \{0\}$ to $L_1 \subset \mathbb{R}^3 \times \{1\}$ if

  $\partial F = (-) L_0 \amalg L_1$ (we will ignore orientations and just think of the $L_0, L_1$ as sets of points)

Categories we talked about:

① Link: ( links $S^1 \hookrightarrow \mathbb{R}^3$, link cobordisms $F \hookrightarrow \mathbb{R}^3 \times [0,1]$ )

not up to isotopy!

(Why?)

up to isotopy in 4D, but with the boundaries fixed! "rel $\partial$"

(relative to the boundary)

⚠ We were studying link isotopy classes; we now study links.

② Diag = category of link diagrams built to exactly describe Link, but in a way we can work with.

We'll think about working in Link but mention when subtleties come up.

Easier cat that we'll actually work a lot with:

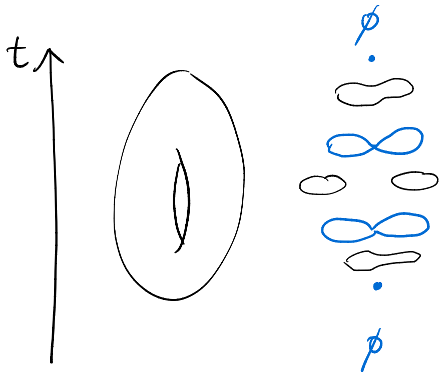
③ Planar circles:

( finite collections of circles in $\mathbb{R}^2$, (no isotopy),

2D cobordisms in $\mathbb{R}^3$ rel $\partial$! )

# Decomposing cobordisms

If we see $\mathbb{T}^2$, a cobordism b/w $\emptyset$ and $\emptyset$, we can reposition it in $\mathbb{R}^3 \times [0,1]$ until, when we scan through, the time slices look like

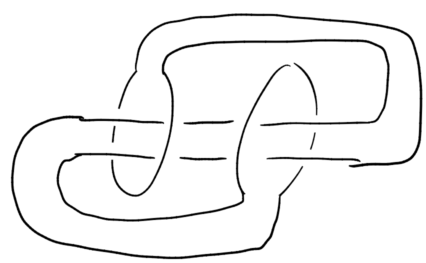


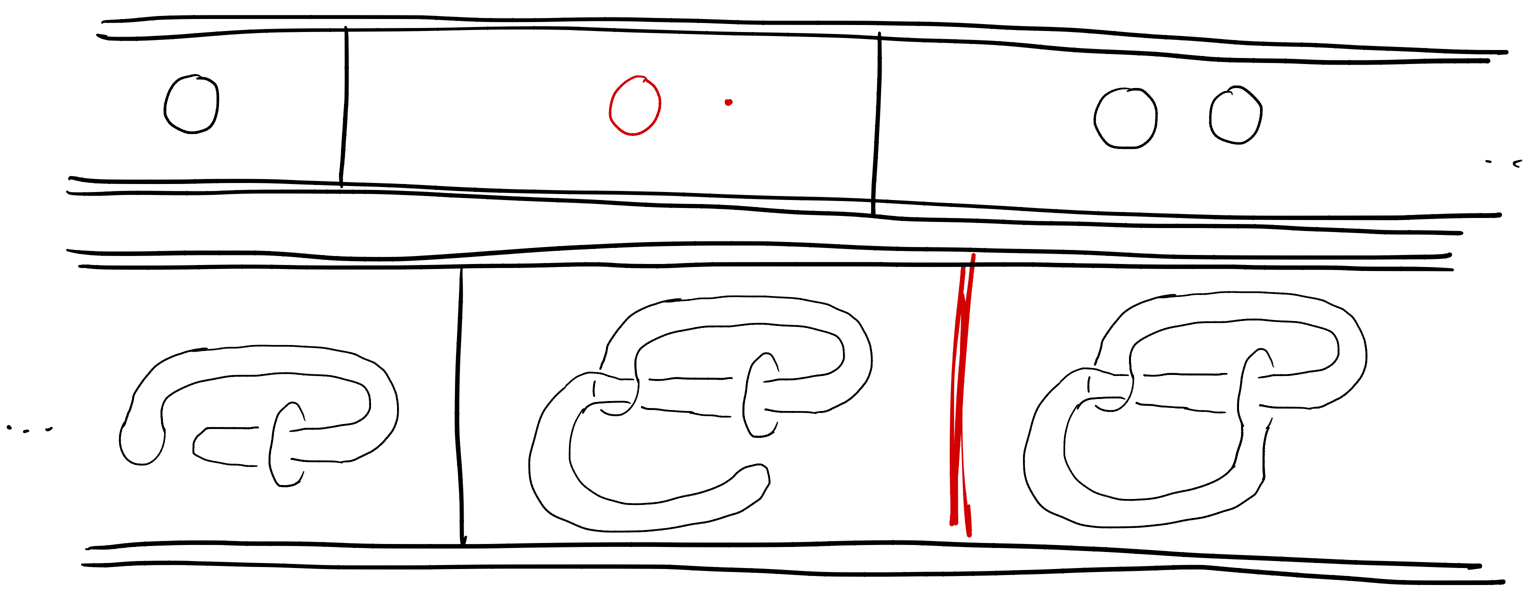

Blue: Critical levels, where topology is changing

Fact. By Morse Theory, we can always decompose our cobordisms into a sequence of movie frames, where a critical moments capture the topology of the level sets changing.

eg. Stevedore Knot $6_1$. Let's look at a cobordism $U \to 6_1$:

draw enough frames for a reader to see what's going on; but make sure you know and indicate where the crit. frames are!





## Subtle point

Movies of Reidemeister moves are critical in the Diag category but not in Link — there are no critical changes in the topology of the surface embedded in 4D!

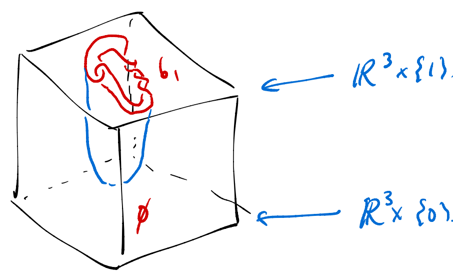
Q. What is this surface $F: U \to 6_1$?

Orientable, 2 boundary components, $\chi = 0$.

$\Rightarrow$ Annulus!

Corollary $K = 6_1$ is a slice knot:

It bounds a disk in 4D, ie




However, $g_3(6_1) = 1$:

$\chi(F) = 5 - 6 = -1$

$\chi(\hat{F}) = 5 - 6 + 1 = 0$

$\Rightarrow \hat{F} \cong T^2$


* alternating diagram $\Rightarrow$ Seifert's algorithm gives a minimal genus Seifert surface.

defn. The 4-ball genus $g_4(K)$ of a knot $K$ is the min genus of slice surfaces $(F, \partial F) \hookrightarrow (B^4, \partial B^3 = S^3)$ for $K$, ie where $K = \partial F \subset S^3$.

(Unpack this defn.)

So for $6_1$, $g_4(K) \lneq g_3(K)$!

In general, we know $g_4(K) \leq g_3(K)$. (Why?)