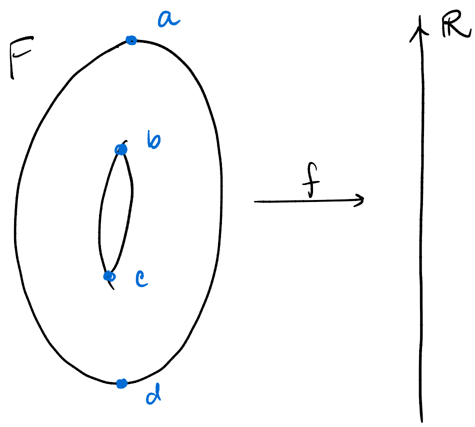
LECTURE 18

Morse Theory ideas, Euler characteristic
Knot Concordance Group

These notes cover topics discussed in Wednesday/Friday's lectures that were not covered in the Lecture 17 notes.

Some ideas from Morse Theory

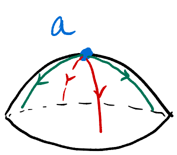


- f is a height function.
- The level sets of f change gradually from

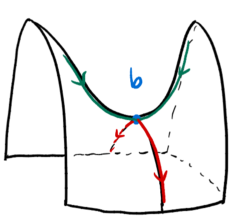
$$\emptyset \to \bigcirc \to \bigcirc\bigcirc \to \bigcirc \to \emptyset$$

with not-1-manifolds occurring only at 4 heights, at 4 critical points.

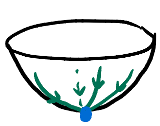
- These are the points where molten chocolate flowing downward on the surface of the donut "stays still".



At the critical point a, chocolate flows away from a in all directions; since F is 2-dimensional, we view these directions as spanned by the red and green dimensions shown.
We say a has "2 unstable dimensions".



At b (and c), there is 1 stable dimension (green) where chocolate flows down toward b (or c).
and 1 unstable dimension, where chocolate flows away from b (or c).



At d, all directions are stable, ie chocolate flows toward d from all directions.
There are 2 stable dimensions.

defn. The index of a critical point p is

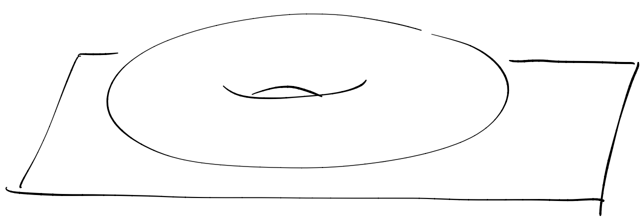
$$\text{index}(p) = \#\text{ unstable dimensions at } p.$$

If the height function is sufficiently nice, ie all the critical points occur at different heights, then we can use the indices of critical points to compute the Euler characteristic of a surface:

$$\chi(F) = \sum_{i=0}^{\infty} (-1)^i \,\#\{\text{critical points } p \text{ with index}(p) = i\}$$

eg. For F above, $\chi(F) = 1 - 2 + 1 = 0$, which agrees with $\chi = 2 - 2g$!

eg. Example of a bad height function:



Donut sitting flat on a table.

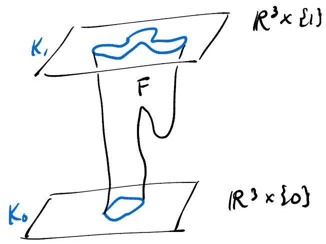
The movie is $\emptyset \rightsquigarrow$ suddenly a whole circle $\rightsquigarrow$ suddenly 2 circles

Too sudden! (not continuous!)

The Knot Concordance Group.

<u>defn.</u> Two knots K_0, K_1 are <u>concordant</u> if there exists a cylinder $F \subset \mathbb{R}^3 \times [0,1]$ such that $\partial F = K_0 \amalg K_1$, where $K_i \subset \mathbb{R}^3 \times \{i\}$ for $i = 0, 1$.

<u>Cartoon</u>



(F is embedded in 4D space and can be very complicated)

<u>defn.</u> The knot concordance group $\mathcal{C} = \{\text{Knots in } \mathbb{R}^3\} / \sim$ where $K_0 \sim K_1$ iff K_0 is <u>concordant</u> to K_1.

eg. $6_1 \sim U$ (see Lect 17 notes)

<u>Fact</u> The connected sum operation on knots "descends" to knot concordance equivalence classes, ie.
If $K \sim K'$ and $J \sim J'$, then $K \# J \sim K' \# J'$.

Thus $\#$ gives us a binary operation on $\mathcal{C}$:
$[K] \cdot [J] = [K \# J]$.

<u>Claim</u> $(\mathcal{C}, \cdot)$ is a group!

<u>defn</u> A group is a set G with a binary operation $\cdot$ such that

① $\exists$ an element $e \in G$ such that $\forall g \in G$, $e \cdot g = g \cdot e = g$.

(This is the identity element.)

② $\forall g \in G$, there is an element $g^{-1} \in G$ such that

$$g \cdot g^{-1} = g^{-1} \cdot g = e.$$

(g^{-1} is the <u>inverse</u> of g, and may or may not be g itself)

③ The operation $\cdot$ should be associative.

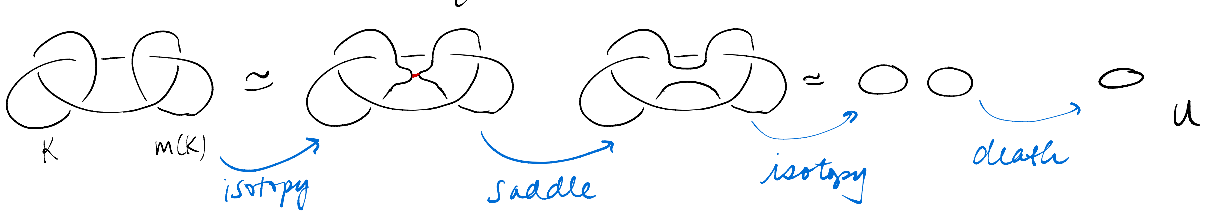
To see that $(\mathcal{C}, \cdot)$ forms a group, note that

- $[U] \in \mathcal{C}$ is the identity element ($K \# U$ is isotopic to K $\forall K$, and isotopy traces out a cylinder in spacetime)
- $\#$ is associative.

It remains to show that every $[K]$ has an inverse. We will show that $[m(K)] = [K]^{-1}$.

We need to show $K \# m(K)$ is concordant to the unknot.

Let's see how to do this via an example; we can then imagine how to extend this argument to all knots.



Let F be the surface this movie depicts.
It has $\chi(F) = 0$ (1 death - 1 saddle) and 2 boundary components.

Convince yourself that this is a cylinder! $\Rightarrow K \# m(K) \sim U$.
eg. cap off both ∂ cpnts $\rightsquigarrow \chi(\hat{F}) = 1 - 1 + 2 = 2 = 2 - 2g \Rightarrow g = 0$.