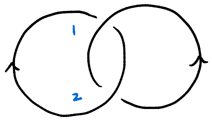
Lecture 19

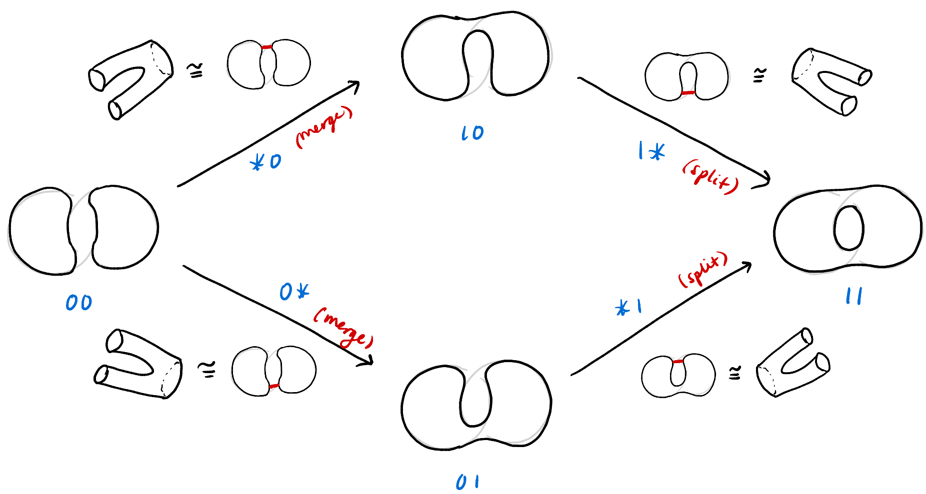
Intro to Khovanov Homology as TQFT

Toward using the categories "Link" and "Planar Circles" to get a knot invariant called Khovanov Homology

↳ defined/discovered by Mikhail Khovanov in 1999 while he was a postdoc here at Davis!

Cube of resolutions





Objects lie at the vertices
Morphisms lie along the edges,
- edges correspond to incrementing exactly one bit from $0 \to 1$.

To pass to linear algebra (ie vector spaces), we go through a TQFT:

> A TQFT is an assignment (functor)
>
> (n-manifolds, (n+1)-dim'l cobordisms) $\longrightarrow$ (Vector spaces, linear maps)
>
> that preserves structure. (details omitted here.)

In our case, we have a TQFT $\mathcal{F}_{Kh}$: Link $\longrightarrow$ $\mathrm{Vect}_{\mathbb{R}}$.

To define $\mathcal{F}_{Kh}$, we will actually work with planar diagrams, ie we need to assign

- unlinks (@ vertices of cube) $\rightsquigarrow$ vector spaces
- cobordisms (merge/split) $\rightsquigarrow$ linear maps.

On Objects

Let $V = \mathbb{R}\langle v_+, v_- \rangle$

v_+, v_- are names of basis vectors, representing two "states" a circle can have:

$\bigcirc^+$ or $\bigcirc^-$

We define $\mathcal{F}_{Kh}(\bigcirc) = V$

View as $\mathcal{F}_{Kh}(\bigcirc) = \mathbb{R}\langle \bigcirc^+, \bigcirc^- \rangle$

For unlink with n components, we assign

$$\mathcal{F}_{Kh}(\underbrace{\circ \cdots \circ}_{n}) = V^{\otimes n} = \underbrace{V \otimes V \otimes \cdots \otimes V}_{n}$$

Tensor product of vector spaces

Tensor product basically models disjoint union:

eg. $\overset{+}{\bigcirc}\ \overset{-}{\bigcirc} \overset{\mathcal{F}_{Kh}}{\rightsquigarrow} v_+ \otimes v_-$

Key fact $3v_+ \otimes 4v_- = 12(v_+ \otimes v_-)$

Numbers can pass through tensors $\otimes$!

Analogy If $f(x)$ and $g(y)$ are single-variable polynomials, their product $f(x)g(y)$ is a 2-variable polynomial whose "monomials" are of the form $x^n y^m$, and numbers can be moved to the front:

eg. $f(x) = 3x \qquad g(y) = 4y$

$$\Rightarrow f(x)g(x) = 3x \cdot 4y = 12 \cdot xy.$$

eg. $\mathcal{F}_{Kh}(\bigcirc\ \bigcirc) = \mathbb{R}\langle \overset{+}{\bigcirc}\overset{+}{\bigcirc}, \overset{+}{\bigcirc}\overset{-}{\bigcirc}, \overset{-}{\bigcirc}\overset{+}{\bigcirc}, \overset{-}{\bigcirc}\overset{-}{\bigcirc} \rangle$

$$= \mathbb{R}\langle v_+ \otimes v_+, v_+ \otimes v_-, v_- \otimes v_+, v_- \otimes v_- \rangle$$

4-dim'l real vector space, where these basis vectors are orthogonal to each other.

On Morphisms

We only need to discuss the linear maps associated to merge and split right now.

First Pass (Just the definition)

merge $m : V \otimes V \longrightarrow V$ as a matrix:

$$\left.\begin{array}{l} v_+ \otimes v_+ \longrightarrow v_+ \\ v_+ \otimes v_- \longrightarrow v_- \\ v_- \otimes v_+ \longrightarrow v_- \\ v_- \otimes v_- \end{array}\right\} \text{means...} \qquad \begin{array}{c|cccc} & v_+\otimes v_+ & v_+\otimes v_- & v_-\otimes v_+ & v_-\otimes v_- \\ \hline v_+ & 1 & 0 & 0 & 0 \\ v_- & 0 & 1 & 1 & 0 \end{array}$$

split $\Delta : V \longrightarrow V \otimes V$ as a matrix:

$$\left.\begin{array}{l} v_+ \longrightarrow v_+ \otimes v_- \\ v_+ \longrightarrow v_- \otimes v_+ \\ v_- \longrightarrow v_- \otimes v_- \\ v_+ \otimes v_+ \end{array}\right\} \text{means...} \qquad \begin{array}{c|cc} & v_+ & v_- \\ \hline v_+\otimes v_+ & 0 & 0 \\ v_+\otimes v_- & 1 & 0 \\ v_-\otimes v_+ & 1 & 0 \\ v_-\otimes v_- & 0 & 1 \end{array}$$

Second Pass (Where did these maps come from?!?)

Replace $v_+ \rightsquigarrow$ "1" and $v_- \rightsquigarrow$ "X" where $1, X \in \mathbb{R}[X] / X^2 \sim 0$

Then m is just multiplication, ie

$m(\alpha \otimes \beta) = \alpha\beta$ as polynomials.

Calculation:

$v_+ \otimes v_+ = 1 \otimes 1 \xmapsto{m} 1 = v_+$

$v_+ \otimes v_- = 1 \otimes X \xmapsto{m} X = v_-$

$v_- \otimes v_+ = X \otimes 1 \xmapsto{m} X = v_-$

$v_- \otimes v_- = X \otimes X \longmapsto X^2 \sim 0.$

> * Don't worry if this doesn't sink in.
>
> This is just "why" these seemingly random maps were chosen.