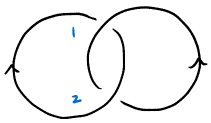
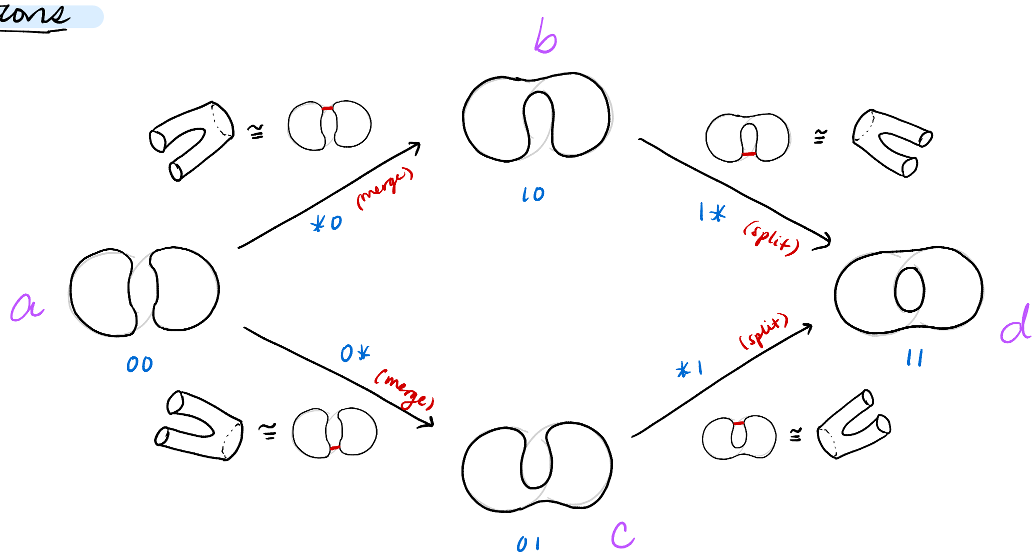
LECTURE 20

<u>Recall</u> Our example started last time:

<u>Cube of resolutions</u>





From the feedback, seems that "tensor products" are a bit daunting. Let's ignore that for now and just label the resolutions as in purple above.

We assign vector spaces to the complete resolutions at the vertices of the cube. The "bases elements" are all the different labellings

$\ell : \{\text{circles in the resolution}\} \longrightarrow \{+,-\}$

eg. a  gets assigned a 4D vector space generated (spanned) by

a_{++} $\quad a_{+-}$ $\quad a_{-+}$ $\quad a_{--}$

ie all possible labelings $\pm$ of the circles in resolution a.

Our cube of resolutions becomes a cube of vector spaces:

$$\begin{tikzcd}
 & \underset{10}{\langle b_+, b_- \rangle} \arrow[dr, "1*"] & \\
\underset{00}{\langle a_{++}, a_{+-}, a_{-+}, a_{--} \rangle} \arrow[ur, "*0"] \arrow[dr, "0*"'] & & \underset{11}{\langle d_{++}, d_{+-}, d_{-+}, d_{--} \rangle} \\
 & \underset{01}{\langle c_+, c_- \rangle} \arrow[ur, "*1"'] &
\end{tikzcd}$$

We now define the linear maps along the "edges" of the cube.

- The linear map corresponding to the "merge" map $*0$ is a matrix

  $A : \langle a_{++}, a_{+-}, a_{-+}, a_{--} \rangle \longrightarrow \langle b_1, b_2 \rangle$

$$\left.\begin{tikzcd}[row sep=small]
a_{++} \arrow[rr] & & b_+ \\
a_{+-} \arrow[rr, bend left] & a_{-+} \arrow[r, bend right] & b_- \\
a_{--} & &
\end{tikzcd}\right\} \overset{\text{this means}}{\Longrightarrow} \quad A = \begin{array}{c} \\ b_+ \\ b_- \end{array} \begin{array}{c} \begin{array}{cccc} a_{++} & a_{+-} & a_{-+} & a_{--} \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \end{bmatrix} \end{array}$$

- The matrix for $0*$ is the same, but is a different map because the codomain of the function is a different vector space.

- The linear map corresponding to the "split" map "$*1$" is defined as

  $B : \langle c_1, c_2 \rangle \longrightarrow \langle d_{++}, d_{+-}, d_{-+}, d_{--} \rangle$

$$\left.\begin{tikzcd}[row sep=small]
 & & d_{++} \\
c_+ \arrow[r] \arrow[rr, bend left] & d_{+-} & d_{-+} \\
c_- \arrow[rr] & & d_{--}
\end{tikzcd}\right\} \overset{\text{this means}}{\Longrightarrow} \quad B = \begin{array}{c} \\ d_{++} \\ d_{+-} \\ d_{-+} \\ d_{--} \end{array} \begin{array}{c} \begin{array}{cc} c_+ & c_- \end{array} \\ \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \end{array}$$

- The matrix for "$1*$" is the same (except that we will need to add a sign later, after we talk about chain complexes.)

Let's do an example.

e.g. $\mathcal{C} = \quad 0 \longrightarrow \underline{\mathbb{R}} \xrightarrow[A_0]{\begin{bmatrix}1\\-1\end{bmatrix}} \mathbb{R}^2 \xrightarrow[A_1]{\begin{bmatrix}1&1\\1&1\end{bmatrix}} \mathbb{R}^2 \xrightarrow[A_2=0]{} 0$

underline means this is C^0

<u>Check that this is a chain complex:</u>

The composition $\quad 0 \longrightarrow \mathbb{R} \xrightarrow{A_0} \mathbb{R}$ is obviously 0.

Same with $\quad \mathbb{R}^2 \xrightarrow{A_1} \mathbb{R}^2 \longrightarrow 0$.

$A_1 A_0 = \begin{bmatrix}1&1\\1&1\end{bmatrix}\begin{bmatrix}1\\-1\end{bmatrix} = \begin{bmatrix}1-1\\1-1\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix}$ is the 0 map $\underline{\mathbb{R}} \longrightarrow \underset{(\text{at } C^2)}{\mathbb{R}^2}$ indeed.

<u>Compute the homology:</u>

At C^0: $\quad \ker(A_0) = 0 \quad (\mathrm{im} = 0 \text{ as well}) \quad \Rightarrow H^0(\mathcal{C}) = \{\vec{0}\}$.

At C^1: $\quad \ker(A_1) = \langle e_1 - e_2 \rangle$

because $e_1 - e_2 = \begin{bmatrix}1\\0\end{bmatrix} - \begin{bmatrix}0\\1\end{bmatrix} = \begin{bmatrix}1\\-1\end{bmatrix}$ and $\begin{bmatrix}1&1\\1&1\end{bmatrix}\begin{bmatrix}1\\-1\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix}$

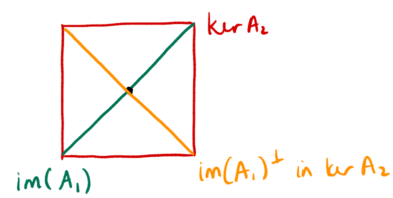
$\mathrm{im}(A_0) = \langle e_1 - e_2 \rangle$ as well $\quad = \left\langle \begin{bmatrix}1\\-1\end{bmatrix} \right\rangle$

$\Rightarrow H^1(\mathcal{C}) = \{\vec{0}\}$

At C^2: $\quad \ker(A_2) = \mathbb{R}^2 = \langle e_1, e_2 \rangle$

$\mathrm{im}(A_1) = \left\langle \begin{bmatrix}1\\1\end{bmatrix} \right\rangle = \langle e_1 + e_2 \rangle$.

$\Rightarrow H^2(\mathcal{C}) = \langle e_1 - e_2 \rangle \cong \mathbb{R}$!



So, we report that

$$H^i(\mathcal{C}) \cong \begin{cases} \mathbb{R} & i = 2 \\ 0 & \text{otherwise} \end{cases}$$

Back to our Hopf link:

$$\begin{tikzcd}[column sep=huge, row sep=large]
 & \underset{10}{\langle b_+, b_- \rangle} \arrow[dr, "1*", "{\begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}}"'] & \\
\underset{00}{\langle a_{++}, a_{+-}, a_{-+}, a_{--} \rangle} \arrow[ur, "{\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \end{bmatrix}}", "*0"'] \arrow[dr, "0*", "{\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \end{bmatrix}}"'] & & \underset{11}{\langle d_{++}, d_{+-}, d_{-+}, d_{--} \rangle} \\
 & \underset{01}{\langle c_+, c_- \rangle} \arrow[ur, "*1", "{\begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}}"'] &
\end{tikzcd}$$

The Khovanov chain complex is built from the cube of vector spaces (and linear maps) as follows:

we need to add a − sign in front of the map along the edge $1*$. (we'll discuss this more later.)

$$C^0 = \langle a_{++}, a_{+-}, a_{-+}, a_{--} \rangle$$

$$C^1 = \langle b_+, b_- \rangle \oplus \langle c_+, c_- \rangle = \langle b_+, b_-, c_+, c_- \rangle$$

combine the vector spaces at vertices of the cube where the binary strings have i ones to form C^i

$$C^2 = \langle d_{++}, d_{+-}, d_{-+}, d_{--} \rangle$$

Chain complex:

$$\langle a_{++}, a_{+-}, a_{-+}, a_{--} \rangle \xrightarrow{\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \end{bmatrix}} \langle b_+, b_-, c_+, c_- \rangle \xrightarrow{\left[\begin{array}{cc|cc} 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{array}\right]} \langle d_{++}, d_{+-}, d_{-+}, d_{--} \rangle$$

You will compute the homology of this chain complex on HW08.