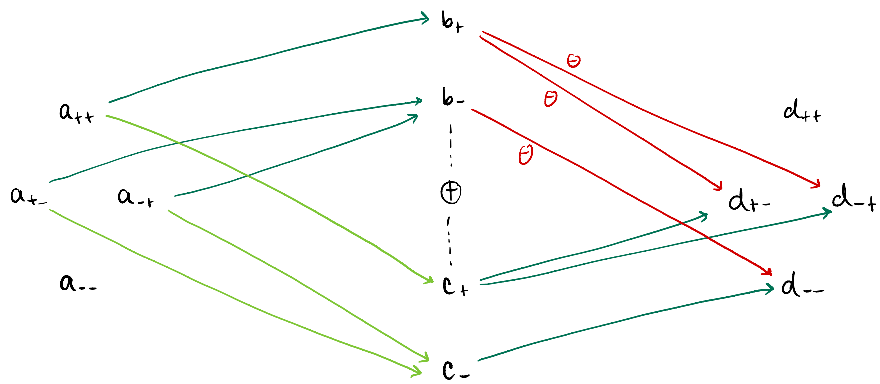
Lecture 21

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \end{bmatrix} \qquad\qquad \begin{bmatrix} 0 & 0 \\ -1 & 0 \\ -1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\langle a_{++}, a_{+-}, a_{-+}, a_{--} \rangle \longrightarrow \langle b_+, b_-, c_+, c_- \rangle \longrightarrow \langle d_{++}, d_{+-}, d_{-+}, d_{--} \rangle$$

How I actually compute the homology:



$$\mathcal{C}^0 \longrightarrow \mathcal{C}^1 \longrightarrow \mathcal{C}^2$$

At $\mathcal{C}^0$: $\quad \ker = \langle a_{+-} - a_{-+}, a_{--} \rangle \qquad \mathrm{im} = 0$

$\Rightarrow H^0(\mathcal{C}) = \langle a_{+-} - a_{-+}, a_{--} \rangle \qquad$ (2-dim'l)

At $\mathcal{C}^1$: $\quad \ker = \langle b_+ + c_+, b_- + c_- \rangle \qquad \mathrm{im} = \langle b_+ + c_+, b_- + c_- \rangle$

$\Rightarrow H^1(\mathcal{C}) = 0$

At $\mathcal{C}^2$: $\quad \ker = \langle d_{++}, d_{+-}, d_{-+}, d_{--} \rangle \qquad \mathrm{im} = \langle d_{+-} + d_{-+}, d_{--} \rangle$

$\Rightarrow H^2(\mathcal{C}) \cong$ orthogonal complement of im in ker

$= \langle d_{++}, d_{+} - d_{-+} \rangle \qquad$ (2 dim'l)

Therefore the Khovanov Homology of the Hopf link is

2-dim'l in H^0 and H^2, and 0-dim'l elsewhere.

(There's more information from the second grading; we'll discuss this next week.)

Aside: Gaussian Elimination

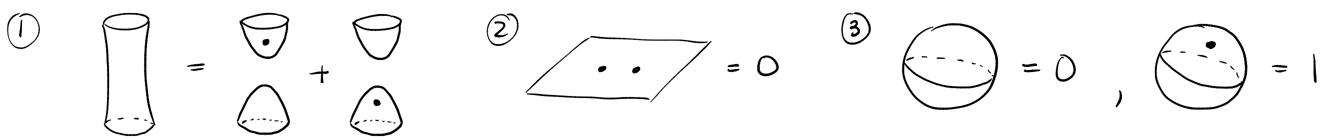
> Recall that Gauss(ian) Elimination is an algorithm that uses row operations to make a matrix upper triangular.
>
> If the Khovanov chain complex were really complicated, I would use an abstract version of Gauss Elimination to compute the homology.

Let's now see how we use topology to determine the merge and split maps.

New category: Bar-Natan Category BN related to "planar circles"

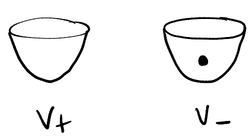
$Ob(BN)$ = same as Ob(planar circles), ie finite collections of circles in $\mathbb{R}^2$

$Mor(BN)$ = dotted cobordisms, subject to the following relations:



Recall that we associate a 2D vector space $\langle v_+, v_- \rangle$ to a single planar circle ⬭.

Let's replace v_+, v_- with cobordisms from $\phi \longrightarrow$ ⬭ :

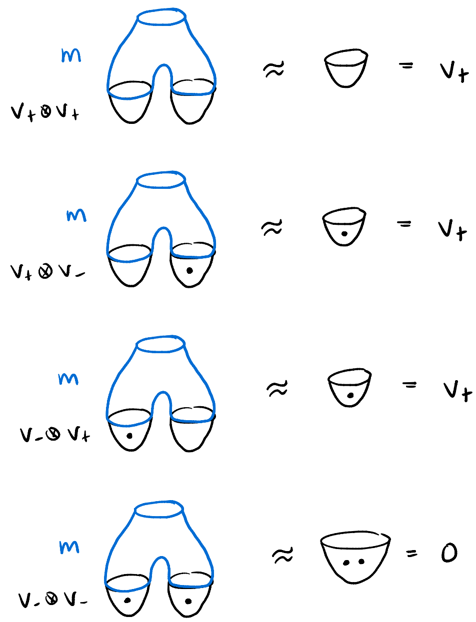


"•" indicates "−"
(or, multiplication by X in $\mathbb{R}[X]/(X^2=0)$)

Then, for example, the vector space associated to ⬭⬭ is generated by

We can now use these bases to determine the m, Δ maps:

Let's apply the merge cobordism to each basis element of  :



We actually usually use the dual vector space and evaluations of closed surfaces to compute the m and Δ matrices.

Dual vector spaces

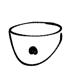
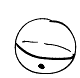
$V = \langle e_1, e_2 \rangle$ $\qquad$ $V^* = \langle e_1^*, e_2^* \rangle$ $\qquad$ $e_i^* : \mathbb{R}^2 \longrightarrow \mathbb{R}$

$$e_i^*(e_j) = \begin{cases} 1 & \text{if } i=j \\ 0 & \text{otherwise} \end{cases}$$

In terms of vectors, e_i^* is e_i^T, because

$$e_i^T e_j = e_i \bullet e_j = \begin{cases} 1 & \text{if } i=j \\ 0 & \text{otherwise} \end{cases}$$

In our case, if $v_+ =$ , then v_+^* must be , since  $= 1$

Similarly, $v_- =$ , so $v_-^* =$ , since  $= 1$.

On the other hand, $v_+^* v_- =$  $= 0$ and $v_-^* v_+ =$  $= 0$.

The matrix for the merge map can be obtained using the following chart: