LECTURE 22: KHOVANOV HOMOLOGY USING LINEAR ALGEBRA OVER $\mathbb{R}$

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Abstract. In these lecture notes, we define Khovanov homology over $\mathbb{R}$ using familiar concepts from linear algebra. We introduce the concepts of chain complexes, their homology, and gradings on chain complexes. We also discuss the relationship between Khovanov homology and the Jones polynomial.

1. Introduction

Khovanov homology is an invariant of links $L \subset S^3$ defined by Khovanov in 2000 [Kho00]. It is computed from a planar diagram D of a link L, by resolving all the crossings in two different ways, in the same way the Kauffman bracket polynomial is computed.

In this lecture, we work with vector fields over $\mathbb{R}$ coefficients. As a "3+1"-dimensional topological quantum field theory, Khovanov homology assigns ($\mathbb{Z} \oplus \mathbb{Z}$-graded) vector spaces to links, and linear maps between to cobordisms between links. We will not discuss the maps assigned to cobordisms.

2. Algebraic Background

We first review some algebraic background.

2.1. Linear algebra basics.

We first review some linear algebra, mostly to set notation and terminology.

Let V and W be real vector spaces of dimensions n and m, respectively. Let f be a *linear transformation* $f : V \to W$. If we choose bases $\{v_1, v_2, \ldots, v_n\}$ and $\{w_1, w_2, \ldots, w_m\}$ for V and W respectively, then we can write f as an $m \times n$ matrix F with real entries.

The *image* of f, denoted $\mathrm{im}\,(f)$, is the subspace of W given by

$$\mathrm{im}\,(f) = \{w \in W \ : \ f(v) = w \text{ for some } v \in V\}.$$

The *kernel* of f, denoted $\ker(f)$, is the subspace of V given by

$$\ker(f) = \{v \in V \ : \ f(v) = 0.\}$$

If $U \subset V$ is a subspace, then the *orthogonal complement* of U is the subspace of V

$$U^\perp = \{v \in V \ : \ \forall u \in U, v \perp u\}.$$

In other words, $v \in U^\perp$ if and only if $v \cdot u = 0$ for all $u \in U$.

Remark. The subspace $U^\perp$ can be identified with the quotient vector space V/U. We choose to work with orthogonal complements to keep our discussion concrete.

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2.2. **Homology of chain complexes.**

Definition. A *chain complex* $\mathcal{C}$ is a sequence of vector spaces $\{C_i\}$ and linear maps $\{d_i\}$

$$\cdots \xrightarrow{d_{i-2}} C_{i-1} \xrightarrow{d_{i-1}} C_i \xrightarrow{d_i} C_{i+1} \xrightarrow{d_{i+1}} \cdots$$

such that for all i, $d_{i+1} \circ d_i = 0$. The C_i are called the *chain groups*, and the d_i are called the *differentials*.

The condition that $d_{i+1} \circ d_i = 0$ is often written in shorthand as $d^2 = 0$. This is equivalent to the requirement that for all i,

$$\mathrm{im}\,(d_i) \subset \ker(d_{i+1}).$$

If this condition holds, then for every i, we can define the *homology of* $\mathcal{C}$ at *homological grading* i to be

$$H_i(\mathcal{C}) = \frac{\ker(f_i)}{\mathrm{im}\,(f_{i-1})} \cong (\mathrm{im}\,(f_{i-1}))^{\perp} \subset \ker(f_i),$$

the orthogonal complement of $\mathrm{im}\,(f_{i-1})$ inside $\ker(f_i)$.

Quite often in low-dimensional topology we work with vectors spaces with a *distinguished basis*. We call these basis vectors *generators* for the chain groups.

Definition. Let $(\mathcal{C}, d)$ be a chain complex with a set of distinguished generators G. A $\mathbb{Z}$*-grading* gr on $\mathcal{C}$ is an assignment $\mathrm{gr} : G \to \mathbb{Z}$ of integers to the distinguished generators of $\mathcal{C}$ such that there is a uniform $r \in \mathbb{Z}$, such that if the image of $g \in G$ under the differential d is

$$d(g) = \sum_i \alpha_i g_i,$$

then $\mathrm{gr}(g_i) = \mathrm{gr}(g) + r$ for all i.

In other words, the differential shifts the grading by a uniform amount everywhere in the complex.

3. The Khovanov chain complex

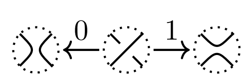
The Khovanov chain complex for an oriented link L is constructed from an oriented diagram D for the link. In this section, we define the Khovanov complex and discuss why $d^2 = 0$. We generally follow Bar-Natan's fantastic introduction to Khovanov homology [BN02]. The reader is encouraged to refer to Section 4 for a concrete example.

Let D be an oriented diagram for L with n_+ positive crossings and n_- negative crossings. Let $n = n_+ + n_-$ be the total number of crossings. Choose any ordering on the crossings, i.e. label the n crossings from 1 to n.

3.1. **Cube of resolutions.**

We first form the cube of resolutions as follows. The 2^n binary strings in the Cartesian product $\{0,1\}^n$ are the vertices of an n-dimensional cube with side length 1 in $\mathbb{R}^n$. If vertices u and v are connected by an edge, then, up to swapping u and v, it must be that they different in only one digit, i.e. $v_i = 1$, $u_i = 0$, and $v_j = u_j$ for all $j \neq i$. In this case, we write $u \prec_1 v$. Each binary string u has a weight $|u| = \sum_{i=1}^n u_i$; if $u \prec_1 v$, then $|v| = |u| + 1$.

Organize this cube of vertices and edges by placing all vertices with the same weight in the same column, with columns organized from weight 0 on the far left to weight n on the far right. Draw lines indicating edges $u \to v$ whenever $u \prec_1 v$.

Each crossing can be smoothed in two different ways: 

At the vertex u, we associate the complete resolution D_u of D given by smoothing the ith crossing according to u_i, for each i.

Label the edge $u \prec_1 v$ by replacing the digit where u and v disagree with an asterisk: $u_1 u_2 \ldots u_{i-1} * u_{i+1} \ldots u_n$. For example, if $u = 010$ and $v = 011$, then the edge between them is labeled $01*$.

3.2. **Chain groups and gradings.** We first associate vector spaces to each vertex D_u of the cube of resolutions. Let Z_u denote the set of planar circles D_u. The *Khovanov chain group* $\mathrm{CKh}(D_u)$ at vertex u is *generated by* the set of labelings of the circles by $\pm$, i.e.

$$\mathrm{CKh}(D_u) = \mathbb{R}\langle\{l : Z_u \to \{+,-\}\}\rangle.$$

So, if there are k circles in D_u, then $\mathrm{CKh}(D_u)$ is a 2^k-dimensional vector space.

Let g be a generator at vertex u, i.e. g is a labeling $g : Z_u \to \pm$.

- The *homological grading* of g is
$$\mathrm{gr}_h(g) = |u| - n_-.$$
- The *quantum grading* of g is
$$\mathrm{gr}_q(g) = |u| + p(g) + n_+ - 2n_-,$$
where $p(g) = (\#+) - (\#-)$ in the labeling g.

The Khovanov chain group at homological grading i is

$$\mathrm{CKh}^i(D) = \bigoplus_{i=|u|-n_-} \mathrm{CKh}(D_u) = \mathbb{R}\langle\{g \;:\; \mathrm{gr}_h(g) = i\}\rangle.$$

3.3. **Differentials.** Let $u \to v$ be an edge in the cube of resolutions. There are two cases:

(1) $|Z_u| = |Z_v| + 1$, in which case two circles in Z_u *merge* to become one in Z_v, or
(2) $|Z_u| = |Z_v| - 1$, in which case one circle in Z_u *splits* to become two circles in Z_v.

In Case 1, we associate the following map of vector spaces. Suppose circles $x_1, x_2 \in Z_u$ merge to become circle $y \in Z_v$. Then the differential associated to $u \to v$, denoted $d_{u\to v}$ for now, is induced by the following linear map:

$$\begin{aligned} ++ &\mapsto + \\ +-, -+ &\mapsto - \\ -- &\mapsto 0. \end{aligned}$$

Here, the sequence of signs (e.g. "$+-$") indicates the labels on x_1, x_2, in that order. This linear map is *extended by identity* on the other circles, meaning that all other circles keep the same label. For example, if $u = 010$ and $v = 011$, and $Z_u = \{x_1, x_2, x_3\}$ and $Z_v = \{y_1, y_2\}$ where x_1 and x_2 merge to become y_1, then the generator $(+-)+ \in \mathrm{CKh}(D_u)$ is sent to $-+ \in \mathrm{CKh}(D_v)$.

The differential $d_i : \mathrm{CKh}^i(D) \to \mathrm{CKh}^{i+1}(D)$ is the sum of the differentials

$$d_i = \sum_{u \prec_1 v,\ i=|u|-n_-} \epsilon_{u\to v} d_{u\to v}$$

where $\epsilon \in \{\pm 1\}$ is -1 to the power of the number of 1s appearing before the asterisk in the label for the edge $u \to v$ in the cube of resolutions. For example, a differential component $01*$ from $u = 010$ to $v = 011$ should have an overall minus sign.

Remark. One can show that, without this sign on the edges, each face of the cube of vector spaces and linear maps commutes, i.e. if we have $u \prec_1 v, v' \prec_1 w$, the composition of the maps associated to $u \to v \to w$ and $u \to v' \to w$ are the same. Then sum of these two maps would then be twice the map $u \to v \to w$. Adding this sign ensures that either 1 or 3 of the edges of this face of the cube has a minus sign, so that d^2 from $u \to w$ is indeed 0, and we have a chain complex.

One can check that the Khovanov differential increases homological grading by 1 and preserves quantum grading. Since quantum grading is preserved, the Khovanov complex is actually a collection of smaller chain complexes, one for each quantum grading:

$$\mathrm{CKh}(D) = \bigoplus_{i,j\in\mathbb{Z}} \mathrm{CKh}^{i,j}(D),$$

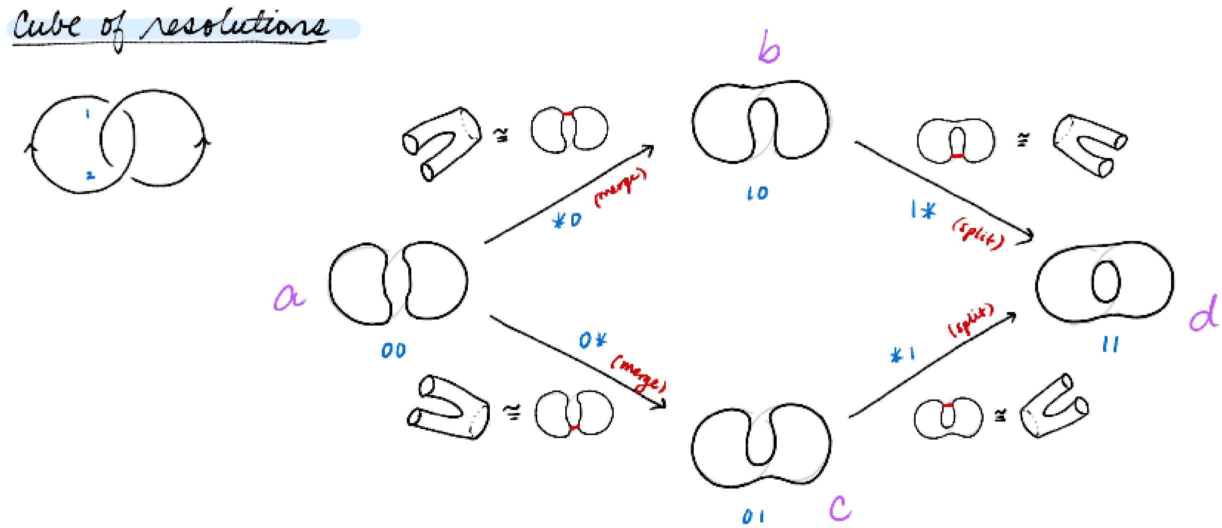


FIGURE 1. The cube of resolutions for the Hopf link, from Lecture 20.

where $\mathrm{CKh}^{i,j}(D)$ is the chain group at homological grading i and quantum grading j.

Remark. There is a convention in homological algebra where, if a chain complex has differentials that increase homological grading, as in our case, it is considered a *cochain complex* that computes *cohomology*. In this sense, Khovanov homology as described here is actually *Khovanov cohomology*.

4. Example: Hopf Link

Let D be the oriented diagram in the top left of Figure 1. Note that $n_+ = 2$ and $n_- = 0$; in particular, the homological grading of generators at any vertex u is just $|u|$.

The four complete resolutions are given labels a, b, c, d in order to distinguish their generators. To avoid confusion, we will use the symbol ∂ for the Khovanov differential.

For the resolution D_{00}, we label the circle on the left first, then the right; for D_{11}, we label the outer circle first, then the inner. With these orderings chosen, our chain groups are as follows:

- $\mathrm{CKh}^0(D) = \mathbb{R}\langle a_{++}, a_{+-}, a_{-+}, a_{--}\rangle$, with the generators in quantum gradings $4, 2, 2, 0$ respectively.
- $\mathrm{CKh}^1(D) = \mathbb{R}\langle b_+, b_-, c_+, c_-\rangle$, with the generators in quantum gradings $4, 2, 4, 2$, respectively.
- $\mathrm{CKh}^2(D) = \mathbb{R}\langle d_{++}, d_{+-}, d_{-+}, d_{--}\rangle$, with the generators in quantum gradings $6, 4, 4, 2$, respectively.

The only edge with a minus sign is $1*$. The differentials are determined by these maps on generators (basis vectors):

- $a_{++} \mapsto b_+ + c_+$
- $a_{+-}, a_{-+} \mapsto b_- + c_-$
- $-b_+, c_+ \mapsto d_{+-} + d_{-+}$
- $-b_-, c_- \mapsto d_{--}$.

All other generators are mapped to 0.

The homology groups are as follows:

- The homology at homological grading $\mathrm{gr}_h = 0$ is

$$\mathrm{Kh}^0(D) = \frac{\ker \partial_0}{\operatorname{im} \partial_{-1}} = \frac{\langle a_{+-} - a_{-+}, a_{--}\rangle}{0} \cong \langle a_{+-} - a_{-+}, a_{--}\rangle.$$

Thus $\mathrm{Kh}^{0,0}(L), \mathrm{Kh}^{0,2}(L) \cong \mathbb{R}$.

- One can check that $\ker = \mathrm{im}$ at homological grading $\mathrm{gr}_h = 1$, and so there is no homology here.
- The homology at $\mathrm{gr}_h = 2$ is
$$\mathrm{Kh}^2(D) = \frac{\ker(\partial_2)}{\mathrm{im}\,(\partial_1)} = \frac{\langle d_{++}, d_{+-}, d_{-+}, d_{--}\rangle}{\langle d_{+-} + d_{-+}, d_{--}\rangle} \cong \mathbb{R}\langle d_{++}, d_{+-} - d_{-+}\rangle.$$
Thus $\mathrm{Kh}^{2,6}(L), \mathrm{Kh}^{2,4}(L) \cong \mathbb{R}$.

Remark. Khovanov homology is an invariant of links only up to (bigraded) isomorphism (because there are many different 1-dimensional vector spaces over $\mathbb{R}$, but they are all isomorphic to $\mathbb{R}$. So, $\mathrm{Kh}(D)$ above when we are computing homology, but write $\mathrm{Kh}(L)$ to indicate the isomorphism class of (bigraded) real vector spaces that $\mathrm{Kh}(L)$ must lie in.

5. Relation to Jones Polynomial

Let $\mathcal{C}$ be a bigraded chain complex with a homological grading gr_h and an additional grading gr_q.

Definition. The *graded Euler characteristic* of a $(\mathrm{gr}_h, \mathrm{gr}_q)$-graded chain complex $\mathcal{C}$ is the Laurent polynomial
$$\chi(\mathcal{C}) = \sum_{i,j\in\mathbb{Z}} (-1)^i q^j \dim H^{i,j}(\mathcal{C}) \in \mathbb{Z}[q, q^{-1}].$$

The graded Euler characteristic of the Khovanov homology $\mathrm{Kh}(L)$ of a link L is the unnormalized Jones polynomial $\widetilde{V}_L(q)$ of the link (up to a sign, depending on conventions). In particular, this means that Khovanov homology determines the Jones polynomial, and is therefore at least as power as the Jones polynomial in distinguishing links. In fact, Bar-Natan exhibited examples showing that Khovanov homology is strictly stronger than the Jones polynomial [BN02]. Furthermore, Kronheimer and Mrowka showed that Khovanov homology *detects* the unknot, i.e. any knot K with the same Khovanov homology as the unknot must be the unknot [KM11]. On the other hand, the analogous statement for the Jones polynomial has been open for nearly half a century:

Question. Does the Jones polynomial detect the unknot?

For comparison, there are plenty of examples of nontrivial knots with the same Alexander polynomial as the unknot.

Remark. To see what the state of the area of Khovanov homology looked like circa 2010, see the lecture series by Paul Turner [Tur].[1]

References

[BN02] Dror Bar-Natan. On Khovanov's categorification of the Jones polynomial. *Algebr. Geom. Topol.*, 2:337–370, 2002.

[Kho00] Mikhail Khovanov. A categorification of the Jones polynomial. *Duke Math. J.*, 101(3):359–426, 2000.

[KM11] P. B. Kronheimer and T. S. Mrowka. Khovanov homology is an unknot-detector. *Publ. Math. Inst. Hautes Études Sci.*, (113):97–208, 2011.

[Tur] Paul Turner. A Hitchhiker's Guide to Khovanov Homology, Part I. `https://www.youtube.com/watch?v=sskEujiB8V8`. Date accessed: May 23, 2023.

[1] This remark was added to show you how to cite a Youtube video.