LECTURE 23

lecture 22 notes are on website, typed with full bibliography and figures.

Last week & Monday:

The graded Euler characteristic of Khovanov Homology is the Jones Polynomial.

- $Kh(L) = \bigoplus Kh^{i,j}(L) \qquad gr_h = i,\ gr_q = j$

  $\chi(Kh(L)) = \sum_{i,j} (-1)^i q^j \dim Kh^{i,j}(L) = \tilde{V}(L)$

- We say

  "Khovanov Homology is a categorification of the Jones polynomial"

- Actually, even if we don't compute homology,

  $\chi(Kh(L)) = \chi(CKh(D)) = \tilde{V}(L)$

  We computed the Kauffman bracket from the same cube we used to compute Kh!

  Why? Gauss elimination of the differential cancels dimensions of a vector space in adjacent homological gradings + in the same quantum grading.

Is there a categorification of the Alexander polynomial? (for a knot K?)

Yes. [Ozsváth-Szabó, Rasmussen ~2003] Knot Floer Homology. $\widehat{HFK}(K)$

This is constructed using very complicated geometry and analysis, but there is a nicer algorithm that computes $\widehat{HFK}(K)$ for knot $K \subset S^3$ called Grid Homology.

Today

- presenting knots using grid diagrams
- what are the "Reidemeister moves" in this context?
- grid matrix + Alex polynomial

Planar grid diagrams

<u>defn</u>. A <u>planar grid diagram</u> $\mathbb{G}$ is an $n \times n$ grid in the plane

(ie. the square $[0,n] \times [0,n]$ with grid lines along integer x, y coords)

where n squares are marked X and n squares are marked O, s.t.

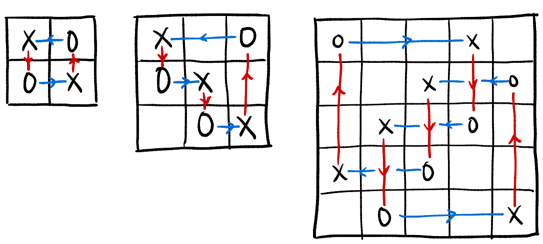
- every row has one X and one O.
- every column has one X and one O
- no square is marked with both an X and an O.

Let $\mathbb{X}$ = the collection of all the Xs, $\mathbb{O}$ = —— all the Os.

The natural number n = grid # of $\mathbb{G}$.

These grid diagrams can record certain knot (or link!) diagrams.
All $L \subset S^3$ have (infinitely many) grid diagrams:

eg.



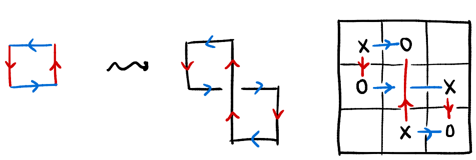
Rules

1. vertically travel from $\mathbb{X} \to \mathbb{O}$
2. horizontally travel from $\mathbb{O} \to \mathbb{X}$
3. vertical arc always passes over the horizontal arc.

<u>Rmk</u>. Minimal grid # is an invariant of L that we know very little about!

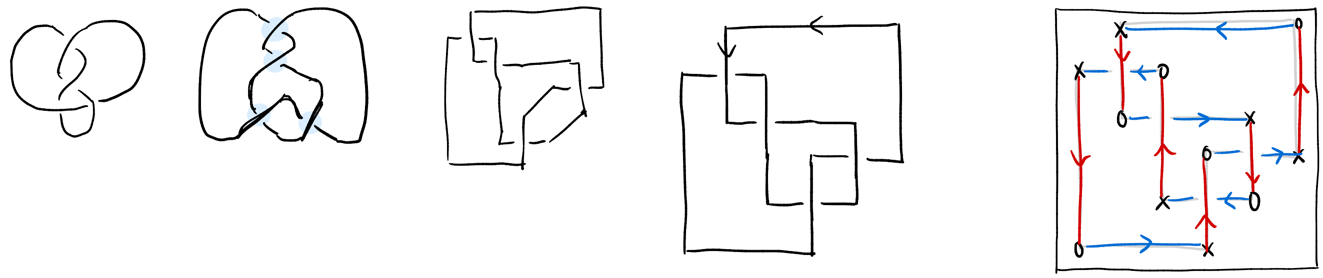
<u>Q</u>. How are grid diagrams for the same L related? <u>Grid moves</u>.

There are many, but here's an interesting one: What does R1 look like?



grid # increased;
we've replaced the SW corner's $\boxed{X}$ with

ex. Draw a planar grid diagram for the F8



ex. Can you draw $\mathbb{G}$ for Hopf link (positively linked?)

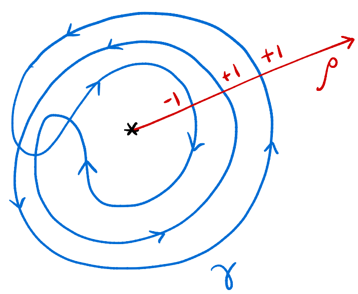
Aside Note that $\mathbb{O}$ and $\mathbb{X}$ are just permutations of the set $\{1, \dots, n\}$.

How many different grid diagrams of size $n \times n$ are there?

Toward Alexander polynomial:

Winding Number: (review of algebraic intersection #)

$\mathbb{R}^2 - \{\vec{0}\}$:



winding of γ around $*$

$= \omega_\gamma(*)$

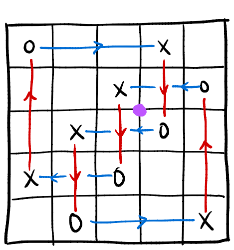
$= I(\rho, \gamma) = +1.$

For a grid diagram $\mathbb{G}$, define the grid matrix $M(\mathbb{G})$ as follows:

$M_{ij} = t^{-w_{\mathbb{L}}(p_{ij})}$ where $p_{ij} = (\underbrace{j-1, n-i}_{})$ lattice point

Annoying convention reasons b/c matrices are indexed $1,1 \rightarrow \downarrow$

eg.

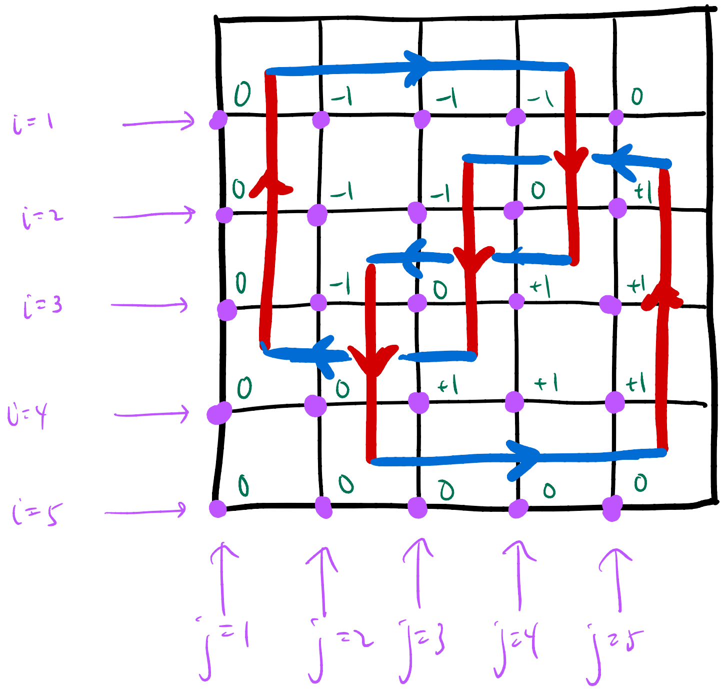


$i = 2, \; j = 4$:

$M_{ij} = M_{2,4}$

$p_{2,4} = (4-1, 5-2) = (3,3)$

$w_{\vec{K}}(p_{2,4}) = 0.$



$P_{ij} = (j-1,\, n-i)$

- $j \in \{1, \dots, 5\} \Rightarrow j-1 \in \{0, 4\}$
- $i \in \{1, \dots, 5\} \Rightarrow n-i \in \{0, 4\}$

$$M = \begin{bmatrix} 1 & t & t & t & 1 \\ 1 & t & t & 1 & t^{-1} \\ 1 & t & 1 & t^{-1} & t^{-1} \\ 1 & 1 & t^{-1} & t^{-1} & t^{-1} \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$

defn. $D_G(t) = \underbrace{\varepsilon(G)}_{\text{sign } \in \pm 1} \cdot \underbrace{\det(M(G)) \cdot (t^{-1/2} - t^{1/2})^{1-n}}_{\text{should be } \doteq \Delta_K(t)\,!} \; \underbrace{t^{a(G)}}_{\text{global shift}}$

sign $\in \pm 1$, depends on the permutations $\mathbb{O}$, $\mathbb{X}$

should be $\doteq \Delta_K(t)$!

up to a power of t, sign.

global shift (we'll use $\doteq$ equivalence, so don't worry about this now)

https://www.wolframalpha.com/input?i=%28determinant+of+%5B%5B1%2Ct%2Ct%2Ct%2C1%5D%2C%5B1%2Ct%2Ct%2C1%2Ct%5E%28-1%29%5D%2C%5B1%2Ct%2C1%2Ct%5E%28-1%29%2Ct%5E%28-1%29%5D%2C+%5B1%2C1%2Ct%5E%28-1%29%2Ct%5E%28-1%29%2Ct%5E%28-1%29%5D%2C%5B1%2C1%2C1%2C1%2C1%5D%5D%29*%28t%5E%28-1%2F2%29-t%5E%281%2F2%29%29%5E%28-4%29

Prop. $D_{\vec{L}_+}(t) - D_{\vec{L}_-}(t) = (t^{1/2} - t^{-1/2})\, D_{\vec{L}_0}(t)$