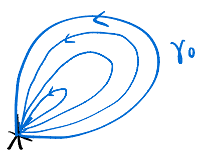
The fundamental group of a topological space

<u>defn.</u> Let X be a topological space with a special "basepoint" $*$.
The fundamental group of $(X,*)$, denoted $\pi_1(X,*)$,
is the group with

- underlying set $= \{\text{hikes from } * \text{ back to } *\}/\sim$

  where $\gamma_0 \sim \gamma_1$ if there is a continuous deformation
  from γ_0 to γ_1 (through paths γ_t, $t \in (0,1)$):

eg. 

In the plane, any loop $\gamma_0 \sim 1$,
the "stay in place" loop.

Eg.
- $\pi_1(S^1,*) = \mathbb{Z}$
- $\pi_1(T^2,*) = \mathbb{Z}\times\mathbb{Z}$
- $\pi_1(S^1 \vee S^1,*) = F_2$

etc.

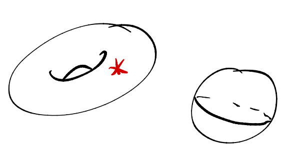
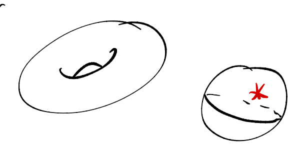
Why do we care about this?

$\pi_1(X)$ gives us a nearly complete prime knot invariant
in the form of an algebraic object!

(So what X do we look at?)

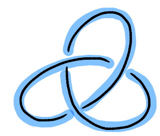
<u>Rmk</u> If X is "path-connected", ie you can drag $*$ to any other point in X, then $\pi_1(X,*)$ doesn't depend on the choice of basepoint $*$.

When does it matter?

eg. $\pi_1\Big($  $\Big) \not\cong \pi_1\Big($  $\Big)$

The Knot (or Link) exterior

defn. For a knot (or link) in $S^3 = \mathbb{R}^3 \cup \{\infty\}$, the 3-sphere, the knot exterior $X_K = S^3 - \nu(K)$ where $\nu(K)$ is a thickened version of K: (neighborhood of K in S^3)



We've been studying knot invariants all quarter, and we generally consider an invariant I to be stronger if it can distinguish more pairs of knots.

The ultimate knot invt would be a "complete knot invariant"

ie if $K_0, K_1 \subset S^3$ such that $I(K_0) = I(K_1) \Rightarrow K_0 \sim K_1$
(ie unique identifier!)

thm. [Gordon-Luecke '89] The knot exterior (homeomorphism class) is a complete knot invariant!

But if we want to be able to apply this, we still need an algebraic invariant:

defn. The knot (or link) group of $K \subset S^3$ is $\pi_1(X_K)$.

thm. [Gordon-Luecke '89] If $\pi_1(X_K) \cong \pi_1(X_J)$ and K and J are both prime knots, then $K \sim J$ or $m(J)$.

Why prime? K = square knot $T \# m(T)$, J = granny knot $T \# T$, T = trefoil

$$\Rightarrow \pi_1(X_K) \cong \pi_1(X_J)\,!$$

thm. [Waldhausen + Gordon-Luecke] The info $\pi_1(\partial \nu(K)) \longrightarrow \pi_1(X_K)$ is a complete knot invariant!

This was used by Ian Agol to prove that ribbon concordance is a partial order on knots within concordance equiv. classes, in Jan. 2022!

[Next: Compute $\pi_1(X_K)$ using grid diagrams!]