LECTURE 25

<u>Announcements</u> ① Final Project: May submit until Sat, 6/3 @ 11:59 pm.

- We'll hear from Camille on Friday!

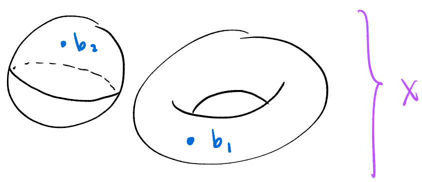
- Presentation: Canvas dropbox
- Expo article: Gradescope
- Outreach: Send to me somehow (email, by hand, part of presentation, etc)

<u>Recall</u> X = top. space. $\pi_1(X,*)$ = fundamental group of X based at the basepoint $*$

$$\{\text{loops starting and ending at } *\}/\sim$$

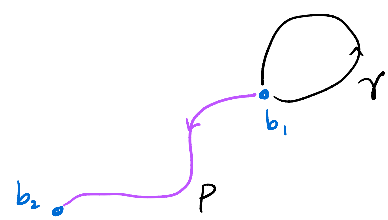
where $\gamma_0 \sim \gamma_1$ if we can deform one path into another continuously

eg.



$\pi_1(X, b_1) \ncong \pi_1(X, b_2)$

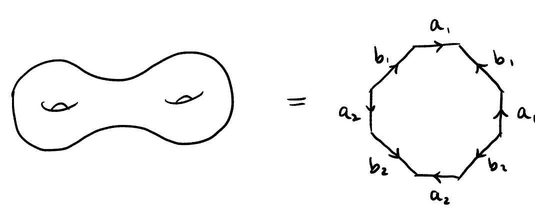
but if X is "path-connected" ie you can drag b_1 to b_2, then $\pi_1(X, b_1) \cong \pi_1(X, b_2)$



$$\gamma \in \pi_1(X, b_1) \longmapsto p\gamma p^{-1} \in \pi_1(X, b_2)$$

So henceforth we will write $\pi_1(X)$ when X is path-connected

eg. Genus 2 surface:



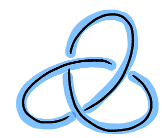
$$\pi_1(\Sigma_2) = \langle a_1, b_1, a_2, b_2 \mid a_1 b_1 a_1^{-1} b_1^{-1} a_2 b_2 a_2^{-1} b_2^{-1} = 1 \rangle$$

eg. Genus g surface Σ_g

$$\pi_1(\Sigma_g) = \langle a_1, b_1, \ldots, a_g, b_g \mid \prod_{i=1}^{g} a_i b_i a_i^{-1} b_i^{-1} = 1 \rangle$$

The Knot (or link) exterior

defn. For a knot (or link) in $S^3 = \mathbb{R}^3 \cup \{\infty\}$, the 3-sphere, the knot exterior $X_K = S^3 - \nu(K)$ where $\nu(K)$ is a thickened version of K: (neighborhood of K in S^3)



We've been studying knot invariants all quarter, and we generally consider an invariant I to be stronger if it can distinguish more pairs of knots.

The ultimate knot invt would be a "complete knot invariant"

ie if $K_0, K_1 \subset S^3$ such that $I(K_0) = I(K_1) \Rightarrow K_0 \sim K_1$
(ie unique identifier!)

thm. [Gordon-Luecke '89] The knot exterior (homeomorphism class) is a complete knot invariant!

But if we want to be able to apply this, we still need an algebraic invariant:

defn. The knot (or link) group of $K \subset S^3$ is $\pi_1(X_K)$.

thm. [Gordon-Luecke '89] If $\pi_1(X_K) \cong \pi_1(X_J)$ and K and J are both prime knots, then $K \sim J$ or $m(J)$.

Why prime? K = square knot, J = granny knot $\qquad T$ = trefoil
$\quad T \# m(T) \qquad\qquad T \# T$

$$\Rightarrow \pi_1(X_K) \cong \pi_1(X_J) \, !$$

thm. [Waldhausen + Gordon-Luecke] The info $\pi_1(\partial \nu(K)) \longrightarrow \pi_1(X_K)$ is a complete knot invariant!

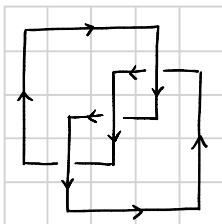
This was used by Ian Agol to prove that ribbon concordance is a partial order on knots within concordance equiv. classes, in Jan. 2022!

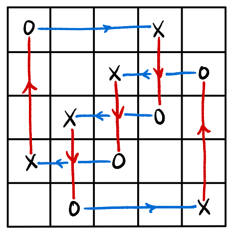
[Next: Compute $\pi_1(X_K)$ using grid diagrams!]

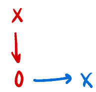
Computing $\pi_1(X_K)$ from a grid diagram of K.

(You can use this for $\pi_1(X_L)$ for a link L as well. We focus on knots because of above motivation.)

eg. $\pi_1(X_K)$ for $K=$ left-handed trefoil







$\pi_1(X_K) = \langle x_1, \dots, x_n \mid r_1, \dots, r_{n-1} \rangle$ $\quad G$ grid number n diagram

generators $= \{ x_1, x_2, x_3, x_4, x_5 \}$

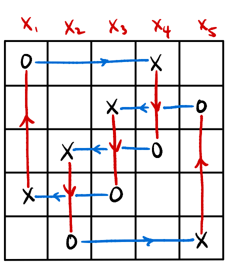
= vertical segments in the grid diagram ("X → O" vertically)

we don't care about the orientations on these arcs.

relations $= \{ r_1, r_2, r_3, r_4 \}$

r_j computed from horizontal lines separating the rows

$= \prod$ generators corr. to the vertical segments that meet the j^{th} horizontal line, in order from left to right.



$\leftarrow r_1 = x_1 x_4$

$\leftarrow r_2 = x_1 x_3 x_4 x_5$

$\leftarrow r_3 = x_1 x_2 x_3 x_5$

$\leftarrow r_4 = x_2 x_5$

$$\pi_1(X_K) = \langle x_1, x_2, x_3, \cancel{x_4}, \cancel{x_5} \mid \underbrace{x_1 x_4}_{x_4 = x_1^{-1}} = x_1 x_3 x_4 x_5 = x_1 x_2 x_3 x_5 = \underbrace{x_2 x_5}_{x_5 = x_2^{-1}} = 1 \rangle$$

$$\pi_1(X_K) = \langle x_1, x_2, x_3 \mid \underbrace{x_1 x_3 x_1^{-1} x_2^{-1}} = x_1 x_2 x_3 x_2^{-1} = 1 \rangle$$

$x_1 x_3 x_1^{-1} x_2^{-1} = 1$

$\Rightarrow x_3 x_1^{-1} x_2^{-1} = x_1^{-1}$

$\Rightarrow x_3 = x_1^{-1} x_2 x_1$

$$\pi_1(X_K) = \langle x_1, x_2 \mid x_1 x_2 (x_1^{-1} x_2 x_1) x_2^{-1} = 1 \rangle$$

$$= \langle x_1, x_2 \mid \underline{x_1 x_2 x_1^{-1}}\, \underline{x_2 x_1 x_2^{-1}} = 1 \rangle$$

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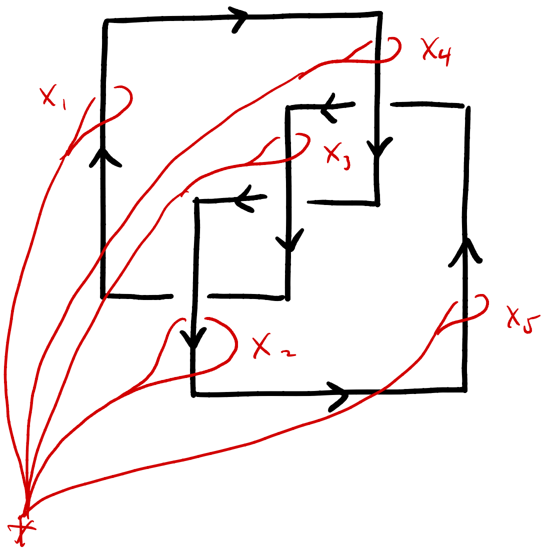
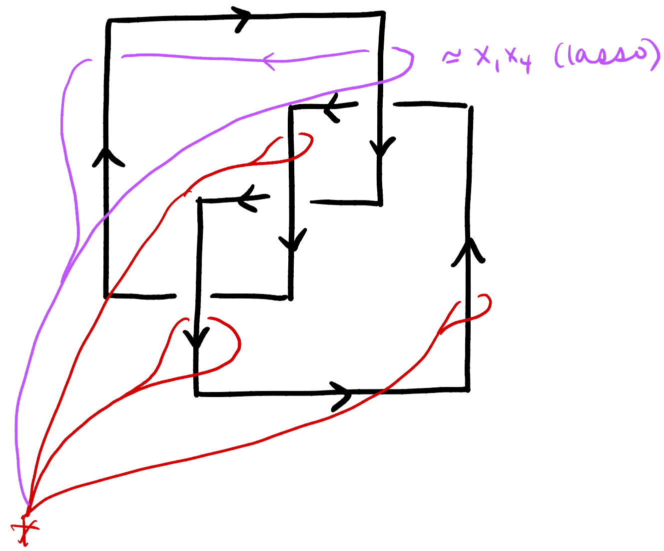
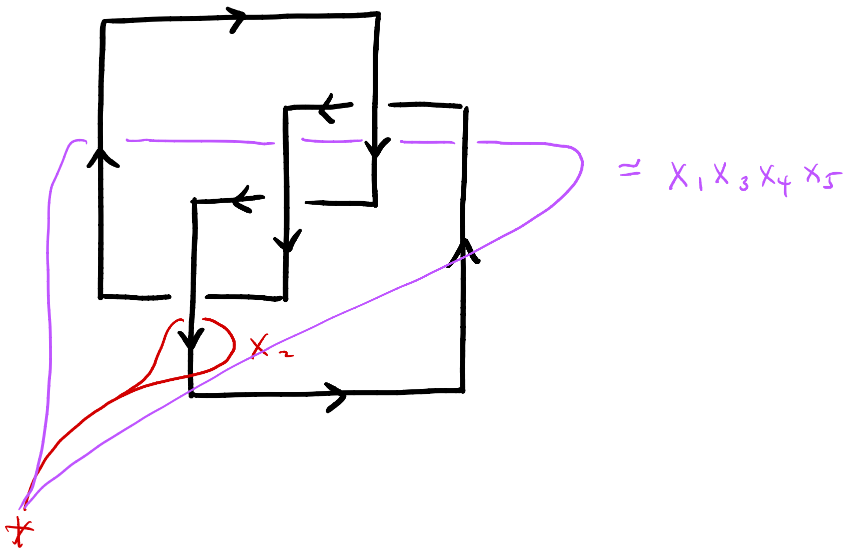
Why does this work?







If you study algebraic topology later (eg. Hatcher's book) then the actual proof comes from Seifert-Van Kampen theorem.

(See also the Wirtinger presentation of $\pi_1(X_K)$ for a usual knot diagram.)