MAT180 HW02

Solutions

Due 4/14/23 at 11:59 pm on Gradescope

Reminder Your homework submission must be typed up in full sentences, with proper mathematical formatting. The following resources may be useful as you learn to use TeX and Overleaf:

- Overleaf's introduction to LaTeX:
`https://www.overleaf.com/learn/latex/Learn_LaTeX_in_30_minutes`
- Detexify:
`https://detexify.kirelabs.org/classify.html`

How will this be graded? You will be graded for accuracy, but also effort. As such, you should attempt every problem, and even if you don't have the full solution, you should write down your thoughts (coherently, please). You are highly encouraged to collaborate with classmates! You must submit your own solution in your own words, though.

If you get stuck for a long time on a problem, talk to people! Discuss with classmates, go to office hours, etc. While it's an important skill to be able to keep at a problem, your time is also valuable; you know what's a reasonable amount of time for yourself to spend on a problem.

Exercise 1

(Ex. 1.2 in the book) Recall that the *crossing number* of a knot K is the minimal number of crossings needed in a diagram representing K. If the crossing number of K is c, then we say K is a *c-crossing knot*.

Prove that there are no two-crossing nontrivial knots.

SOLUTION.

Refer to Figure 1. Suppose we have a minimal two-crossing diagram D of a knot. Up to planar isotopy, we can arrange the two (orange) crossings beside each other. Since we know the crossings are part of the same knot, there must be a blue arc connecting them. Up to planar isotopy and reflection, we can consider just cases. For each case, we then enumerate where the northwest endpoint connects to the rest of the diagram using the red arcs. Finally, we use black arcs to connect up the rest of the endpoints so that the resulting diagram is a knot, i.e. has only one component. This leaves us with the eight cases in Exercise 1. By inspection, all of these are unknots! So, the diagram was not minimal, and we have a contradiction.

Exercise 2

This is a multi-part exercise about the writhe of diagrams.

(a) Compute the writhe of the knot diagram shown in Figure 2.

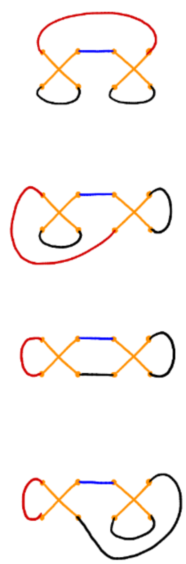


Figure 1: Cases for Exercise 1.

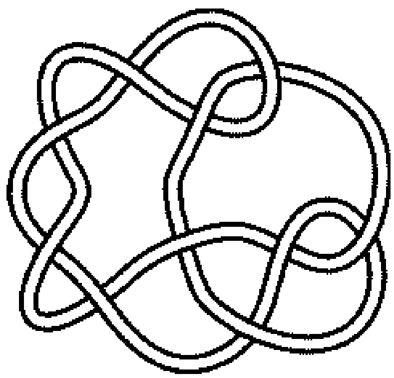
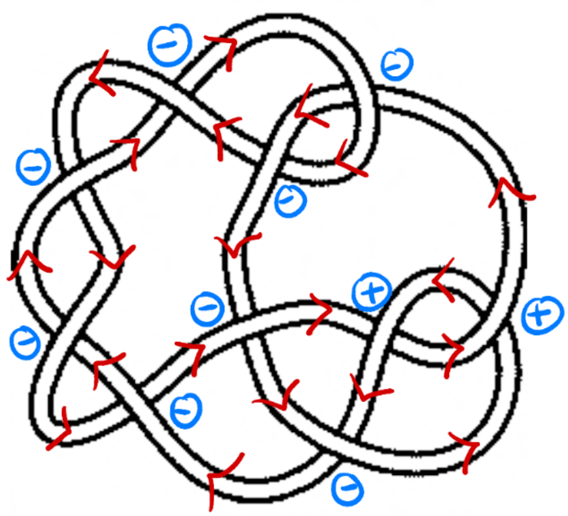


Figure 2: A diagram of the knot 10_{36}, taken from Knot Atlas.

(b) Describe how the writhe of a diagram changes under all the Reidemeister moves. (Note that there are two different Reidemeister 1 and 3 moves.)

SOLUTION.

(a) We choose an orientation on the knot and compute the writhe below:

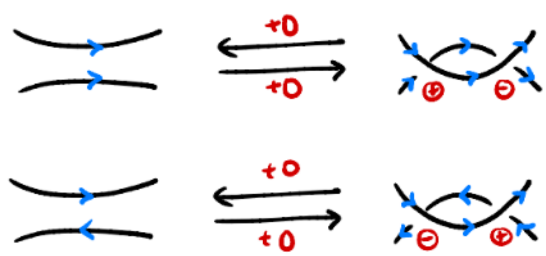


writhe = # ⊕ − # ⊖

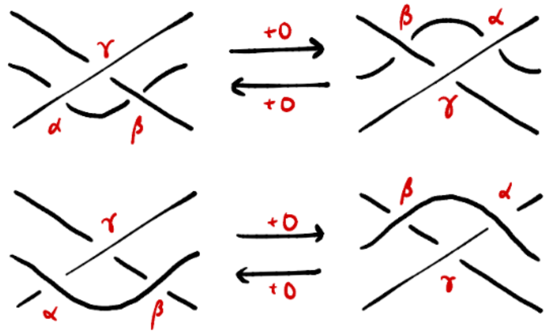
= 2 − 8 = −6

(b) First consider Reidemeister 1. Regardless of the orientation of the strand in the middle of the diagram below, the twisted strand on the left (resp. right) introduces a negative (resp. positive) crossing. The change in writhe is recorded below next to the move they correspond to.

Next, consider Reidemeister 2. The two strands in the local diagram without crossings may be parallel or anti-parallel. But in either case, the two introduced crossings have opposite sign, so the writhe remains unchanged:



Finally, consider Reidemeister 3. Regardless of the orientations of the three strands in each local diagram, each crossing on the local diagram on the left can be matched with a crossing on the diagram on the right:



Therefore the writhe of the overall diagram is also preserved under Reidmeister 3.

Exercise 3

(1.17 in the book) Compute the absolute values of the linking numbers of the two links shown in Figure 1.39 of the book in order to show that they must be distinct links.

Solution.

(Abridged) All crossings in both link diagrams are the same sign. However, two of the crossings on the second link diagram involve only one component of the link, and thus do not count toward the linking number. So, the linking number of the first link is ± 3 while the linking number of the second link is ± 2. Since linking number is an invariant of (2-component) links, these two links are distinct.

Exercise 4

(1.25, 1.27 in the book)

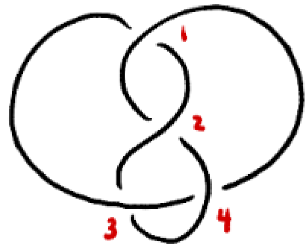
(a) Show that the composition of any knot with a tricolorable knot yields a new tricolorable knot.

(b) Prove that the figure-eight knot (4_1 in the Rolfsen knot table) is *not* tricolorable. Conclude that the figure-eight knot and the trefoil knot are distinct knots.

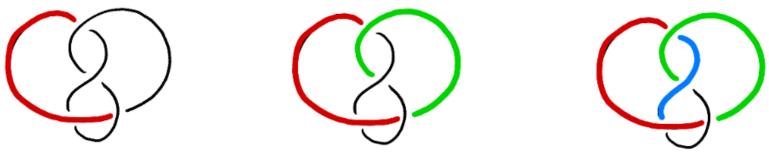
Solution.

(a) Let T be a tricolorable knot and K an arbitrary other knot. Let D_T be a tricolored diagram of T, and D_K a diagram of K. Let c be the color of the strand in D_T at the site of the knot composition; color all strands of D_K by c. Then, the composition $D_T \# D_K$ is tricolored, since (1) at least two colors were used (since D_T was tricolored) and (2) all crossings satisfy the tricoloration requirements (the new ones in D_K are all the same color).

(b) We showed in class that tricolorability is preserved under Reidemester moves. So, we may choose to use the following diagram of the knot 4_1, where we label the crossings for easy reference:



Without loss of generality, we may color the left-most strand red, as seen in the left-most image below:



Now consider the right-most strand. It is relatively straightforward to check that if it is also colored red, then we would be forced to color the whole diagram red, which is not allowed, because tricolorings must use at least two colors. So, we color the right-most strand a different color, green. Looking at crossing 1, this forces the middle strand to be colored the third color, blue.

We now have to decide what to color the bottom strand. On the one hand, crossing 3 forces this strand to be green. At the same time, crossing 2 forces this strand to be red. Therefore we have reached a contradiction[1], and so 4_1 is not tricolorable.

Exercise 5

(1.6, 1.7 in the book)

(a) Show that by changing the crossings from over to under or vice versa, any projection of a knot can be made into the projection of an alternating knot. (This isn't as easy as it might seem. How do you know your procedure will always work?)

(b) In a projection with n crossings, what is the maximum number of crossings that would have to be changed in order to make the knot alternating?

(c) Show that by changing some of the crossings from over to under or vice versa, any projection of a knot can be made into a projection of the unknot.

Solution.

(a) There are two ways I can think of to prove this.

Suppose you accept that (1) any knot diagram has a checkerboard coloring, and that (2) alternating knot diagrams have only one type of sign in their associated signed planar graphs.

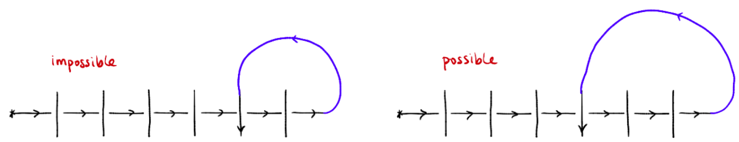
[1] if we had assumed at the beginning of the proof that we were reconstructing a tricoloring of this diagram

Then, given an arbitrary knot diagram D with n crossings, we can formed its signed planar graph Γ_D, which has n edges, each labeled with a $+$ or $-$.

We can then consider all the crossings in D that correspond to a $(-)$-labeled edge in Γ_D and change the crossing. The resulting diagram D' would have a signed planar graph with only $(+)$-labeled edges; therefore D' is alternating.

□

There is another way to prove the statement which does not depend on (1) and (2) above. Here's a sketch of the proof. Once again, let D be an arbitrary knot diagram. Forget the crossing information; we will now start at one point on D and assign crossing information as we travel along the knot. It remains to show that when you return to a crossing whose crossing information you have already decided, that you can traveling through it while maintaining the alternation:



Note that along the blue arc, there are no crossings. Show that the parity of the number of ends (top of vertical sticks in the pictures above) enclosed by the blue strand must be even. Argue that this then ensures that when you arrive at the crossing you've already been to, you can maintain alternation.

(b) Using the planar graph point of view, if D has n crossings, then you at worst need to change $\lfloor \frac{n}{2} \rfloor$ crossings to make the signed planar graph all $+$ or all $-$ labeled.

(c) (Proof sketch) Using a similar idea to the second approach to part (a), you can start at a particular point in D and, as you approach each new crossing, make sure you travel under. As a result, your path in 3D space is always descending, until you return to the point where you started, where you travel vertically upward to connect up the loop. This is clearly the unknot as you can isotop your position at every point during your descent to another vertical line; then your knot is basically a rectangle, which is an unknot.