MAT180 HW03

Solutions

Due ~~4/21/23~~ **Monday, 4/24/23** at 11:59 pm on Gradescope

Reminder Your homework submission must be typed up in full sentences, with proper mathematical formatting. The following resources may be useful as you learn to use TeX and Overleaf:

- Overleaf's introduction to LaTeX:
  `https://www.overleaf.com/learn/latex/Learn_LaTeX_in_30_minutes`

- Detexify:
  `https://detexify.kirelabs.org/classify.html`

How will this be graded? You will be graded for accuracy, but also effort. As such, you should attempt every problem, and even if you don't have the full solution, you should write down your thoughts (coherently, please). You are highly encouraged to collaborate with classmates! You must submit your own solution in your own words, though.

If you get stuck for a long time on a problem, talk to people! Discuss with classmates, go to office hours, etc. While it's an important skill to be able to keep at a problem, your time is also valuable; you know what's a reasonable amount of time for yourself to spend on a problem.

Covered material This homework covers material from §2.4 and parts of §6.1-4 in the book. We will revisit some of the material in §6.1-4 again later in the course, in different contexts.

Exercise 1

In class, we sketched a proof showing that every knot diagram has a checkerboard coloring. Write down a proof of this statement in your own words.

Solution.

(Solution omitted; this is an exercise in digesting and reiterating proofs.)

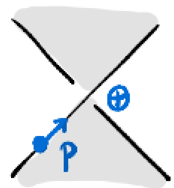
Exercise 2

Show that if a diagram is alternating, then the associated signed planar graph has only one type of sign (either all $+$ or all $-$).

Solution.

Let D be an alternating diagram, and fix a checkerboard coloring c. Pick a point p on a strand in D and orient D so that at p, if an observer were facing forward with respect to the orientation, the region to the right of the observer is shaded. Now observe that when the observer walks through the first crossing, the shaded region switches to the left of the observer. There are now two cases.

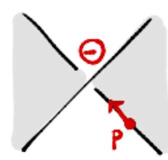
Case 1 The observer walks over the first crossing. Then the checkerboard coloring at this first crossing is a $+$ crossing:



Right before passing under the next crossing (because D is alternating), the observer sees that the region to their left is shaded. Therefore this crossing is also a $+$ crossing.

Continuing in this fashion, the observer passes through the n crossings of D, twice each; at each crossing, if the observer is passing over the crossing, then the checkerboard coloring starts at the right and flips to the left. If the observer is passing under the crossing, then the checkerboard coloring starts at the left and flips to the right. In either case, the sign of the edge in the signed planar graph corresponding to each crossing is $+$.

Case 2 If the observer passes under the first crossing, then we have a similar situation. At the first crossing, the observer passes under the crossing, and the checkerboard coloring flips from the right of the observer to the left:



This means that the (edge in the signed planar graph associated to the) crossing is labeled $-$. At the next crossing, the observer is passing over the crossing, but the checkerboard coloring flips from the left to the right, so the crossing is also labeled $-$. By a similar argument as in Case 1, we see that all the edges in the signed planar graph associated to the pair (D, c) are labeled $-$.

Exercise 3

Using the appropriate skein relations, compute the following polynomials for the figure-eight knot 4_1.

(It might be wise to include a figure for your resolving tree and only type up the part where you report the final polynomial you arrive at, or any polynomial calculations you do to arrive there. You do NOT need to type up the whole resolving tree!)

(a) Jones polynomial $V_K(q)$

(b) Alexander polynomial $\Delta_K(t)$

(c) HOMFLYPT polynomial $P_K(\alpha, z)$

(d) Then, verify that the HOMFLYPT polynomial specializes to the Jones and Alexander polynomials you computed.

Solution.

Let $K = 4_1 = F8$. Below is the resolving tree we used in class. (You may have used a different oriented diagram, but since 4_1 is amphicheiral, we should get the same answers.)

(a) We compute the Jones polynomial using the skein relation for $V_K(q)$ below.

Using the first level of the resolving tree, we get

$$q^{-2}V_K - q^2 V_U = (q - q^{-1})V_{H^-}.$$

Using the second level of the resolving tree, we get

$$q^{-2}V_{U^2} - q^2 V_{H^-} = (q - q^{-1})V_U.$$

In class, we showed that $V_{U^2}(q) = -(q + q^{-1})$; therefore

$$\begin{aligned} q^{-2}(-q - q^{-1}) - q^2 V_{H^-} &= (q - q^{-1}) \\ V_{H_-}(q) &= -(q^{-1} + q^{-5}). \end{aligned}$$

(Note the similarity between $V_{H_-}(q)$ and V_{H_+}.)

Returning to the first level of the resolving tree, we have

$$\begin{aligned} q^{-2}V_K - q^2 &= (q - q^{-1})(-q^{-1} - q^{-5}) = -1 + q^{-2} - q^{-4} + q^{-6} \\ q^{-2}V_K &= q^2 - 1 + q^{-2} - q^{-4} + q^{-6} \\ V_K(q) &= q^4 - q^2 + 1 - q^{-2} + q^{-4}. \end{aligned}$$

(Observe that this polynomial is palindromic and alternating...)

(b) The computation for the Alexander polynomial is similar to the computation of the Jones polynomial. (Computation omitted.)

$$\Delta_K(t) = -t + 3 - t^{-1}$$

(c) From the resolving tree, we know that we will need the HOMFLYPT polynomial of the two component unknot U^2. To do this, we consider the 1-crossing diagram of the unknot. We find that

$$(\alpha - \alpha^{-1})P_U = zP_{U^2}.$$

Therefore,

$$P_{U^2} = \frac{\alpha - \alpha^{-1}}{z}.$$

Using the first level of the resolving tree, we have

$$\begin{aligned} \alpha P_K - \alpha^{-1}P_U &= zP_{H^-} \\ \alpha P_K &= \alpha^{-1}P_U + zP_{H^-}. \end{aligned}$$

To find P_{H^-}, we write

$$\begin{aligned}
\alpha P_{U^2} - \alpha^{-1} P_{H^-} &= z \\
\alpha^{-1} P_{H^-} &= \frac{\alpha^2 - 1}{z} - z \\
P_{H^-} &= \alpha \left(\frac{\alpha^2 - 1}{z} - z \right)
\end{aligned}$$

Substituting this into the original skein relation, we have

$$\begin{aligned}
\alpha P_K &= \alpha^{-1} + z\alpha \left(\frac{\alpha^2 - 1}{z} - z \right) \\
&= \alpha^{-1} + (\alpha^3 - \alpha) - z^2 \alpha \\
P_K &= \alpha^{-2} - 1 + \alpha^2 - z^2.
\end{aligned}$$

To recover the Jones polynomial, we replace $\alpha \mapsto q^{-2}$ and $z \mapsto q - q^{-1}$:

$$\begin{aligned}
\alpha^{-2} - 1 + \alpha^2 - z^2 &\mapsto q^4 - 1 + q^{-4} - (q - q^{-1})^2 \\
&= q^4 - 1 + q^{-4} - (q - q^{-1})^2 \\
&= (q^4 - 1 + q^{-4}) - (q^2 - 2 + q^{-2}) \\
&= q^4 - q^2 + 1 - q^{-2} + q^{-4} \\
&= V_K(q).
\end{aligned}$$

To recover the Alexander polynomial, we replace $\alpha \mapsto 1, z \mapsto t^{1/2} - t^{-1/2}$:

$$\begin{aligned}
\alpha^{-2} - 1 + \alpha^2 - z^2 &\mapsto 1 - 1 + 1 - (t^{1/2} - t^{-1/2})^2 \\
&= 1 - (t - 2 + t^{-1}) \\
&= -t + 3 - t^{-1} \\
&= \Delta_K(q).
\end{aligned}$$

Exercise 4

Prove that for an amphicheiral knot K, the Jones polynomial is palindromic, i.e. $V_K(t) = V_K(t^{-1})$.

Solution.

(The following solution uses the q variable; recall that to obtain the result for the t variable, just set $t = q^2$. You can also prove this using the bracket polynomial; we use the skein relation to illustrate the role of induction in proving facts about recursively defined knot invariants.)

Given any oriented link (diagram) (K, o), let $m(K)$ denote the mirror link (diagram), with the induced orientation. We will first show that for any K,

$$V_{m(K)}(q) = V_K(q^{-1}). \tag{1}$$

Consider the resolving tree $T(K)$ of K. We may assume that the leaves of the tree are unlinks. The resolving tree of $m(K)$ has the same shape; given $D \in V(T(K))$, let D' denote the corresponding node in $V(T(m(K))$; in fact, $D' = m(D)$.

Note that for any (oriented) unlink U^k, Equation 1 holds; this is because up to isotopy, there is only one oriented unlink of k components. In other words, if D is a leaf of $T(K)$, then $V_{D'}(q) = V_D(q^{-1})$.

We now move up the tree $T(m(K))$ by induction; at each vertex (node), we are effectively inducting on the height of the subtree below the node.

Suppose at a non-leaf node, we have a diagram D'. Focusing on a particular crossing c in D', we have either $D' = D'_+$ or $D' = D'_-$. Observe that $D'_\pm = D_\mp$ and $D'_0 = m(D_0)$.

Let's first consider the case where $D = D_+$. We have

$$q^2 V_{D_+})(q) - q^{-2} V_{D_-}(q) = (q - q^{-1}) V_{D_0}(q).$$

By the relationships between the two resolving trees, this is equivalent to

$$q^2 V_{D'_-}(q) - q^{-2} V_{D'_+}(q) = (q - q^{-1}) V_{D'_0}(q).$$

By the induction hypothesis, we have

$$q^2 V_{D_-}(q^{-1}) - q^{-2} V_{D'_+}(q) = (q - q^{-1}) V_{D_0}(q^{-1}).$$

Reorganizing, this is the same as the equation

$$q^{-2} V_{D'_+}(q) - q^2 V_{D_-}(q^{-1}) = (q^{-1} - q) V_{D_0}(q^{-1}).$$

Compare this with Equation 1; this is the same equation if we replace $V_{D'_+}(q)$ with $V_{D_+}(q^{-1})$. In other words, we can solve the two equations for $V_{D'_+}(q)$ and $V_{D_+}(q^{-1})$ and get the same expression; therefore they are the same Laurent polynomial.

The case for $D = D_-$ is essentially the same. Then, Equation 1 holds by induction. Finally, if K is amphicheiral, then $m(K) = K$, so

$$V_K(q) = V_{m(K)}(q) = V_K(q^{-1}).$$