MAT180 HW05

Solutions

Due 5/5/23 at 11:59 pm on Gradescope

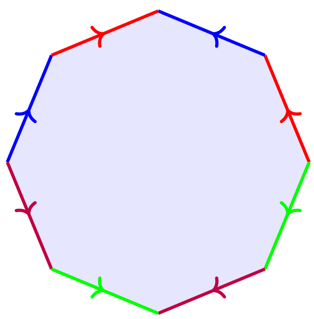
Reminder Your homework submission must be typed up in full sentences, with proper mathematical formatting. The following resources may be useful as you learn to use TeX and Overleaf:

- Overleaf's introduction to LaTeX:
  `https://www.overleaf.com/learn/latex/Learn_LaTeX_in_30_minutes`

- Detexify:
  `https://detexify.kirelabs.org/classify.html`

Exercise 1

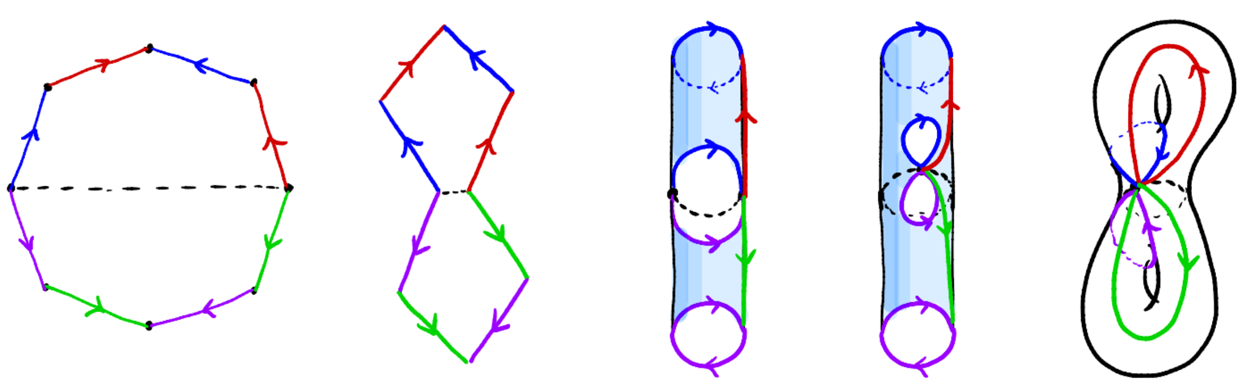
We mentioned in class that a genus-2 orientable surface could be constructed by gluing the edges of an octagon together:



Sketch a series of pictures showing how to see that, after gluing up all the same colored edges according to the orientations shown, you will have the standard picture of a genus 2 surface.

Solution.

Here is one way of visualizing the gluing process:



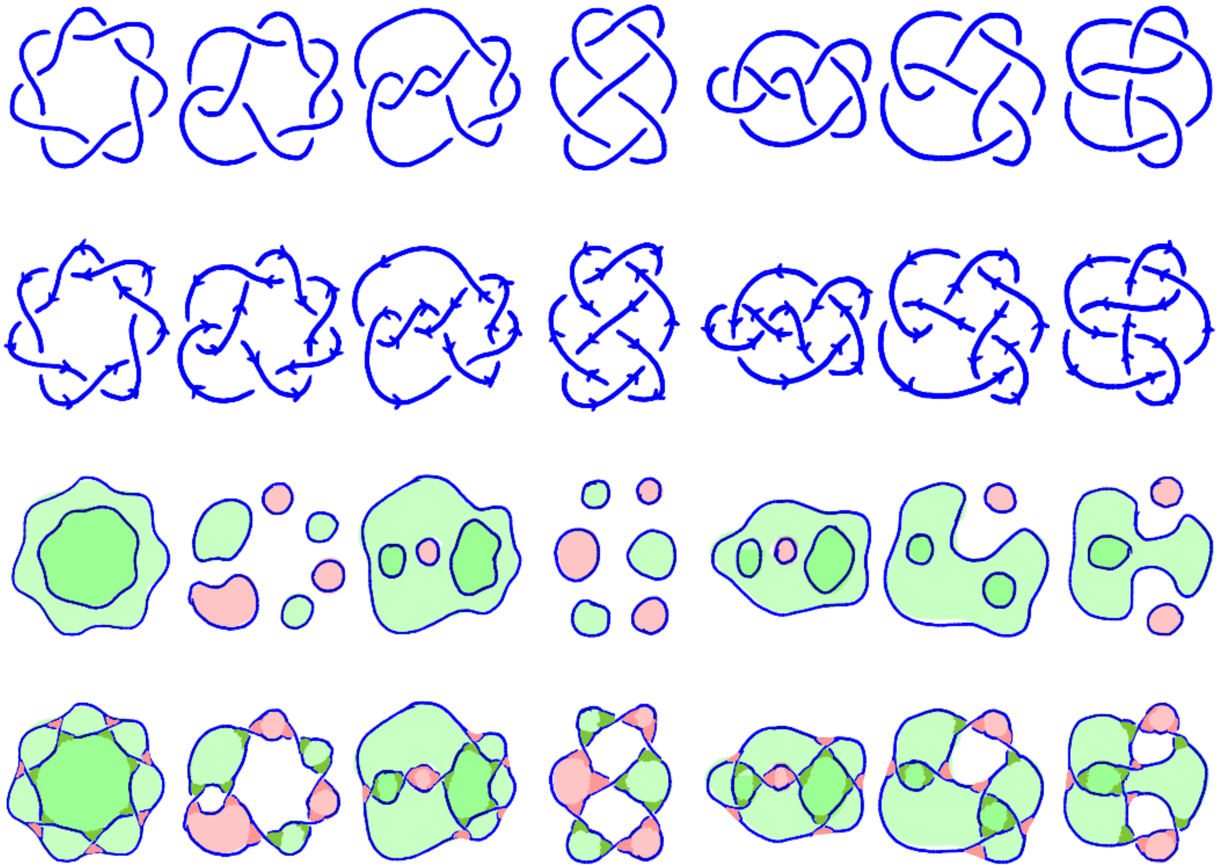


Figure 1: Seifert's algorithm applied to the Knot Atlas diagram for each 7-crossing knot.

Exercise 2

In this exercise, you will explore the 3-genera of 7-crossing knots.

(a) Using Seifert's algorithm, draw Seifert surfaces for all seven 7-crossing knots in the Rolfsen knot table (available in the back of your book, or at `http://katlas.org/wiki/The_Rolfsen_Knot_Table`).

(b) Now compute the genera of the surfaces you drew.

(c) Compare the genera of the Seifert surfaces you drew with the known 3-genera of the 7-crossing knots, which you can access by using KnotInfo: `https://knotinfo.math.indiana.edu/index.php?isdesktop=1`. (At the top of the page, select 7 crossing knots; then find the "`Genus-3D`" property and check the box. Click `Submit` to see the results.)

Solution.

(a) Refer to Figure 1. The first row shows diagrams of the knots 7_1 through 7_7 from the Knot Atlas. In the second row, we have chosen an orientation on each knot. Using these orientations, we draw the Seifert circles in row 3; here, green disks are bounded by CCW Seifert circles, while red disks are bounded by CW Seifert circles. In row 4, we add in the bands to realize the crossings for each knot diagram.

(b) If F is a Seifert surface for a knot K, then F has exactly one boundary component, since $\partial F = K$. Let $\hat{F}$ denote the closed surface we get from capping off this boundary component. If F is constructed from d disks and e bands, then the Euler characteristic of the closed, capped-off surface $\hat{F}$ is

$$\chi(\hat{F}) = d - e + 1 = 2 - 2g(\hat{F}) = 2 - 2g(F).$$

In our case, every diagram has seven crossings, so $e = 7$. We can then solve the equation

$$d - e + 1 = 2 - 2g$$

for g in terms of d:

$$\begin{aligned} d - e + 1 = d - 7 + 1 &= 2 - 2g \\ d - 8 &= -2g \\ g &= 4 - \frac{d}{2}. \end{aligned}$$

From the last row of Figure 1, we obtain the following disk counts, and then compute the corresponding genera:

knot	7_1	7_2	7_3	7_4	7_5	7_6	7_7
# Seifert disks d	2	6	4	6	4	4	4
genus $g(F)$	3	1	2	1	2	2	2

(c) KnotInfo tells us that the Seifert surfaces in Figure 1 are all in fact minimal genus.

Exercise 3

In Section 4.3 of the book, around page 100, Adams proves:

Theorem. The 3-genus of knots is additive, i.e. $g_3(K\#J) = g_3(K) + g_3(J)$.

Read the proof, and then provide an *executive summary* of the proof.

Executive summaries In mathematical writing, proofs must be extremely rigorous, and hence quite long. In research articles, we often first give a shorter summary of the proof of our main results, so that an expert reader can get the main gist of the steps of the proof, without getting bogged down by the details. Your summary should

- be shorter than the actual proof,
- describe all the main steps and insights from the actual proof, including how to resolve any difficulties that arise,
- and convince someone familiar with the field that they could reconstruct the whole proof if given enough time.

Solution.

Note My summary might look very different from yours. Your writing style will change as you read and write more and more mathematics, and my writing style is still evolving (and hopefully improving) as well! As long as you wrote a summary that describes the gist of the argument, you'll receive full (or most) credit.

Below, I use $\mathbb{S}$ to denote a sphere and S for a Seifert surface; Adams uses F for a sphere and S for a Seifert surface, which is counterintuitive for me.

Executive summary As in the statement of the Theorem, let J and K denote knots. Let $g = g_3$ denote 3-ball genus. Adams first argues that $g(J\#K) \leq g(J) + g(K)$, by noting that, if we had minimal genus Seifert surfaces F_J and F_K for J and K respectively, then we can use a band to glue them together to form a new surface $F_\#$ which is a Seifert surface for $J\#K$. Since the genus of this surface is $g(F_J) + g(F_K)$, this is now an upper bound for $g(J\#K)$.

To show that this is actually the minimal genus possible, we let S be any minimal genus Seifert surface for $J\#K$, and isotop the surface until we see the boundary knot $J\#K$ as a clear connected sum. We then choose a sphere $\mathbb{S}$ separating the two components of the connected sum, and isotop it so that the intersection of $\mathbb{S}$ and S consists only of arcs and loops.

In fact, there can only be one arc, since we know that $J\#K = \partial S$ intersects $\mathbb{S}$ at exactly two points. The rest of the intersection $\mathbb{S} \cap S$ consists of loops, all of which bound disks in $\mathbb{S}$. Starting with any "innermost" loop γ, which bounds a disk $D_\gamma \subset \mathbb{S}$, Adams removes loops by modifying the surface S, by cutting S along γ, and then capping off both sides by gluing in two disks that are parallel to D_γ. After throwing out any closed components, we get a modified Seifert surface S' for $J\#K$ that no longer intersects $\mathbb{S}$ along γ. Using Euler characteristic, one can show that this "surgery" operation does not increase genus; that is, $g(S') \leq g(S)$. We repeat this process to remove all loop-type intersections, and therefore the resulting Seifert surface (call it S^*) is of same form as $F_\#$, i.e. obtainable from gluing two Seifert surfaces for J and K together along a band. Therefore $g(S) \geq g(S^*) \geq g(J) + g(K) \geq g(S)$, so all these quantities must be equal.