

MAT180 HW06

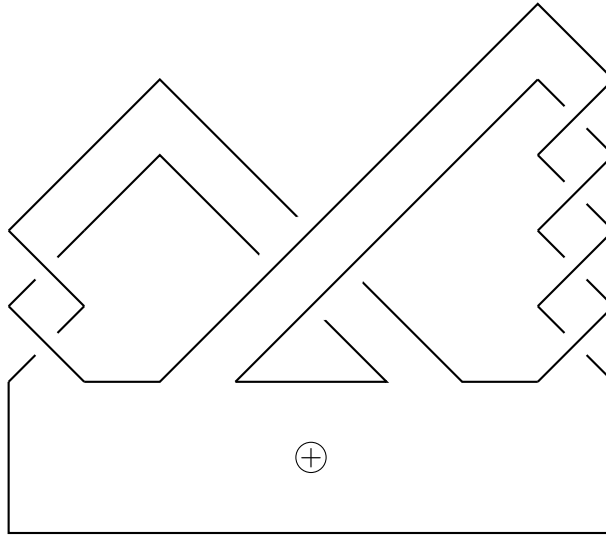
(ADD NAME)

Due 5/12/23 at 11:59 pm on Gradescope

Reminder. Your homework submission must be typed up in full sentences, with proper mathematical formatting.

Exercise 1

Below is a depiction of the Stevedore knot 6_1 , drawn as the boundary of an oriented Seifert surface built out of one disk and two twisted bands:



- Let γ_1 be the counter-clockwise loop that goes through the left band (only, exactly once), and let γ_2 be the counter-clockwise loop going through the right band (only, exactly once). Sketch a diagram of the surface F , the curves γ_1, γ_2 , and their pushoffs γ_1^+, γ_2^+ .
- Construct the Seifert matrix V for this surface.
- Compute $\det(V - tV^T)$; then, if needed, multiply it by t^n for some $n \in \mathbb{Z}$ (possibly negative) so that the powers of t appearing in the Alexander polynomial of 6_1 are balanced around 0.

Exercise 2

Consider the usual diagram of the [Figure 8 in the Rolfsen knot table](#).

- Draw the Seifert surface F resulting from Seifert's algorithm such that 'most' of the surface is facing upward (i.e. you see more of the positive side of F than the negative side). Since there are four crossings, you should have four bands connecting the Seifert disks.

- (b) The surface F is *homotopy equivalent* to a graph with four edges. Draw this graph Γ . Then, contract edges until you have a graph Γ_* with only one vertex; draw this process as a sequence of graphs ending with Γ_* .
- (c) Using your Γ_* , choose a basis $\{\gamma_i\}$ for the first homology $H_1(F)$ of F . (Remember to orient your curves!) Draw these curves on a copy of your surface F ; be careful near the crossings! Draw push-offs $\{\gamma_i^+\}$ in a different color.
- (d) Compute the Seifert matrix V with respect to the data $(F, \{\gamma_i\})$ that you've chosen.
- (e) Compute $\det(V - tV^\top)$ and compare this with the Alexander polynomial of the Figure 8 knot.

Exercise 3

Let V be a matrix corresponding to some data $(F, \{\gamma_i\}_{i=1}^{2g})$, where $g = g_3(F)$. Let \tilde{F} be a stabilization of F , and let \tilde{V} be the Seifert matrix computed from the data $(\tilde{F}, \{\gamma_i\} \cup \{\tilde{\gamma}_1, \tilde{\gamma}_2\})$, where

$$\text{lk}(\tilde{\gamma}_1, \tilde{\gamma}_1^+) = \text{lk}(\tilde{\gamma}_2, \tilde{\gamma}_1^+) = \text{lk}(\tilde{\gamma}_2, \tilde{\gamma}_2^+) = 0 \quad \text{and} \quad \text{lk}(\tilde{\gamma}_1, \tilde{\gamma}_2^+) = 1.$$

Therefore \tilde{V} must have the following form:

$$\tilde{V} = \left(\begin{array}{c|cc} V & & U \\ \hline L & 0 & 1 \\ & 0 & 0 \end{array} \right),$$

where $U \in \mathbb{R}^{2g \times 2}$ and $L \in \mathbb{R}^{2 \times 2g}$.

We want to understand how $\det(\tilde{V} - t\tilde{V}^\top)$ is related to $\det(V - tV)$. This is somewhat difficult in general. However, if we consider the special case where $U = L^\top = 0 \in \mathbb{R}^{2g \times 2}$ (as in the example drawn in class), then

$$\tilde{V} = \left(\begin{array}{c|cc} V & & 0 \\ \hline 0 & 0 & 1 \\ & 0 & 0 \end{array} \right).$$

In this case, it is not difficult to compute $\det(\tilde{V} - t\tilde{V}^\top)$, by following the steps below.

- (a) Write down the matrix $\tilde{V} - t\tilde{V}^\top$ in terms of V .
- (b) Let D denote the bottom right 2×2 block of the matrix $\tilde{V} - t\tilde{V}^\top$. Prove that D is invertible, i.e. its determinant is a nonzero Laurent polynomial (a polynomial in negative, 0, and positive powers of the variable t). Then, find the matrix inverse of D .
- (c) Compute the Alexander polynomial $\det(\tilde{V} - t\tilde{V}^\top)$ by using the following fact from linear algebra, in terms of the polynomial $\det(V - tV^\top)$.

Fact. If $\begin{bmatrix} A & B \\ C & D \end{bmatrix}$ is a block matrix where D is invertible, then

$$\det \left(\begin{bmatrix} A & B \\ C & D \end{bmatrix} \right) = \det(D) \det(A - BD^{-1}C).$$