MAT180 HW06

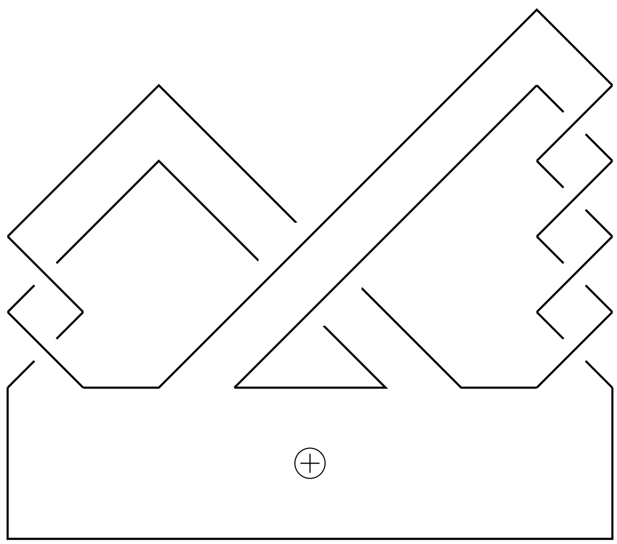
Solutions

Due 5/12/23 at 11:59 pm on Gradescope

Reminder. Your homework submission must be typed up in full sentences, with proper mathematical formatting.

Exercise 1

Below is a depiction of the Stevedore knot 6_1, drawn as the boundary of an oriented Seifert surface built out of one disk and two twisted bands:



(a) Let γ_1 be the counter-clockwise loop that goes through the left band (only, exactly once), and let γ_2 be the counter-clockwise loop going through the right band (only, exactly once). Sketch a diagram of the surface F, the curves γ_1, γ_2, and their pushoffs γ_1^+, γ_2^+.

(b) Construct the Seifert matrix V for this surface.

(c) Compute $\det(V - tV^\top)$; then, if needed, multiply it by t^n for some $n \in \mathbb{Z}$ (possibly negative) so that the powers of t appearing in the Alexander polynomial of 6_1 are balanced around 0.

Solution.

(a) See Figure 1.

(b) Using Figure 1, we compute the following linking numbers:

$$\begin{array}{ll} \mathrm{lk}(\gamma_1, \gamma_1^+) = -1 & \mathrm{lk}(\gamma_1, \gamma_2^+) = 0 \\ \mathrm{lk}(\gamma_2, \gamma_1^+) = -1 & \mathrm{lk}(\gamma_2, \gamma_2^+) = +2 \end{array}$$

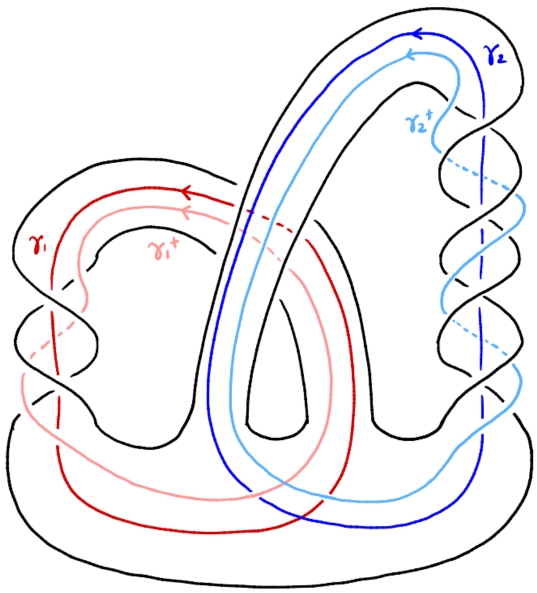


Figure 1: A choice of basis $\{\gamma_1, \gamma_2\}$ for $H_1(F)$, where F is the Seifert surface for 6_1 in Exercise 1.

The Seifert matrix for $(F, \{\gamma_1, \gamma_2\})$ is

$$V = \begin{bmatrix} -1 & 0 \\ -1 & 2 \end{bmatrix}.$$

(c) We compute

$$\det(V - tV^\top) = \det\left(\begin{bmatrix} -1 & 0 \\ -1 & 2 \end{bmatrix} - t\begin{bmatrix} -1 & -1 \\ 0 & 2 \end{bmatrix}\right) = \det\left(\begin{bmatrix} -1+t & t \\ -1 & 2-2t \end{bmatrix}\right)$$
$$= (-1+t)(2-2t) + t = -2t^2 + 5t - 2,$$

which, up to an integer power of t, is equivalent to $\Delta_K(t) = -2t + 5 - 2t^{-1}$, where $K = 6_1$.

Exercise 2

Consider the usual diagram of the Figure 8 in the Rolfsen knot table.

(a) Draw the Seifert surface F resulting from Seifert's algorithm such that 'most' of the surface is facing upward (i.e. you see more of the positive side of F than the negative side). Since there are four crossings, you should have four bands connecting the Seifert disks.

(b) The surface F is *homotopy equivalent* to a graph with four edges. Draw this graph Γ. Then, contract edges until you have a graph Γ_* with only one vertex; draw this process as a sequence of graphs ending with Γ_*.

(c) Using your Γ_*, choose a basis $\{\gamma_i\}$ for the first homology $H_1(F)$ of F. (Remember to orient your curves!) Draw these curves on a copy of your surface F; be careful near the crossings! Draw push-offs $\{\gamma_i^+\}$ in a different color.

(d) Compute the Seifert matrix V with respect to the data $(F, \{\gamma_i\})$ that you've chosen.

(e) Compute $\det(V - tV^\top)$ and compare this with the Alexander polynomial of the Figure 8 knot.

Solution.

(a) See Figure 2.

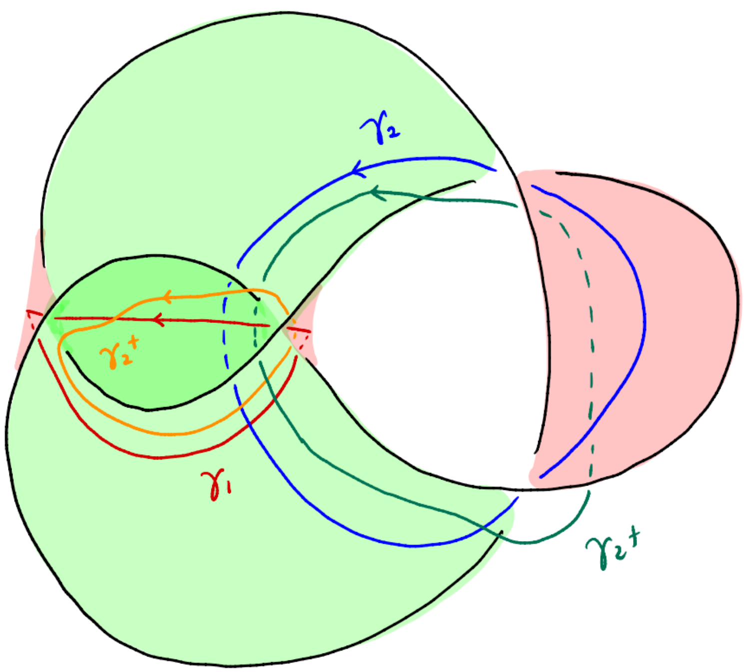


Figure 2: A Seifert surface G for $J = 4_1$ obtained from Seifert's algorithm, along with a choice of basis $\{\gamma_1, \gamma_2\}$ for $H_1(G)$.

(b) See Figure 3.

Figure 3: On the left, the graph Γ where vertices are given by the Seifert disks for the surface G, and edges are given by the attached bands corresponding the crossings of the diagram of J. On the right, a homotopy equivalent graph with just one vertex, and a choice of orientation on the remaining loops.

(c) See Figures 2 and 3.

(d) Using my choice of $(G, \{\gamma_1, \gamma_2\})$, the Seifert matrix is

$$V = \begin{bmatrix} -1 & +1 \\ 0 & +1 \end{bmatrix}$$

(e) We compute:

$$\det(V - tV^\top) = \begin{bmatrix} -1 & 1 \\ 0 & 1 \end{bmatrix} - t\begin{bmatrix} -1 & 0 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} -1+t & 1 \\ -t & 1-t \end{bmatrix}$$
$$= (-1+t)(1-t) + t = -t^2 + 3t - 1,$$

which is equivalent (up to a power of t) to $\Delta_J(t) = -t + 3 - t^{-1}$.

Exercise 3

Let V be a matrix corresponding to some data $(F, \{\gamma_i\}_{i=1}^{2g})$, where $g = g_3(F)$. Let $\tilde{F}$ be a stabilization of F, and let $\widetilde{V}$ be the Seifert matrix computed from the data $(\tilde{F}, \{\gamma_i\} \cup \{\tilde{\gamma}_1, \tilde{\gamma}_2\})$, where

$$\mathrm{lk}(\tilde{\gamma}_1, \tilde{\gamma}_1^+) = \mathrm{lk}(\tilde{\gamma}_2, \tilde{\gamma}_1^+) = \mathrm{lk}(\tilde{\gamma}_2, \tilde{\gamma}_2^+) = 0 \qquad \text{and} \qquad \mathrm{lk}(\tilde{\gamma}_1, \tilde{\gamma}_2^+) = 1.$$

Therefore $\widetilde{V}$ must have the following form:

$$\widetilde{V} = \left(\begin{array}{c|cc} V & \multicolumn{2}{c}{U} \\ \hline \multirow{2}{*}{L} & 0 & 1 \\ & 0 & 0 \end{array}\right),$$

where $U \in \mathbb{R}^{2g\times 2}$ and $L \in \mathbb{R}^{2\times 2g}$.

We want to understand how $\det(\widetilde{V} - t\widetilde{V})$ is related to $\det(V - tV)$. This is somewhat difficult in general. However, if we consider the special case where $U = L^\top = 0 \in \mathbb{R}^{2g\times 2}$ (as in the example drawn in class), then

$$\widetilde{V} = \left(\begin{array}{c|cc} V & \multicolumn{2}{c}{0} \\ \hline \multirow{2}{*}{0} & 0 & 1 \\ & 0 & 0 \end{array}\right).$$

In this case, it is not difficult to compute $\det(\widetilde{V} - t\widetilde{V}^\top)$, by following the steps below.

(a) Write down the matrix $\widetilde{V} - t\widetilde{V}^\top$ in terms of V.

(b) Let D denote the bottom right 2×2 block of the matrix $\widetilde{V} - t\widetilde{V}^\top$. Prove that D is invertible, i.e. its determinant is a nonzero Laurent polynomial (a polynomial in negative, 0, and positive powers of the variable t). Then, find the matrix inverse of D.

(c) Compute the Alexander polynomial $\det(\widetilde{V} - t\widetilde{V}^\top)$ by using the following fact from linear algebra, in terms of the polynomial $\det(V - tV^\top)$.

Fact. If $\begin{bmatrix} A & B \\ C & D \end{bmatrix}$ is a block matrix where D is invertible, then

$$\det\left(\begin{bmatrix} A & B \\ C & D \end{bmatrix}\right) = \det(D)\det(A - BD^{-1}C).$$

Solution.

(a)
$$\widetilde{V} - t\widetilde{V}^\top = \left(\begin{array}{c|cc} V - tV^\top & \multicolumn{2}{c}{0} \\ \hline \multirow{2}{*}{0} & 0 & 1 \\ & -t & 0 \end{array}\right).$$

(b) The determinant of $D = \begin{matrix} 0 & 1 \\ -t & 0 \end{matrix}$ is t, which is nonzero.

(c) In this case, $B, C = 0$, so

$$\det(\widetilde{V} - t\widetilde{V}^\top) = \det(D)\det(A) = t \cdot \det(V - tV^\top).$$

Therefore, up to an overall power of t, $\det(\widetilde{V} - t\widetilde{V}^\top)$ is equivalent to $\det(V - tV^\top)$, so the Alexander polynomials agree.