MAT180 HW07

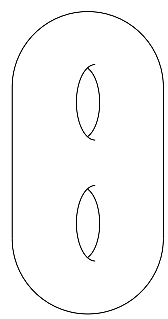
(ADD NAME)

Due 5/19/23 at 11:59 pm on Gradescope

Reminder. Your homework submission must be typed up in full sentences, with proper mathematical formatting.

Exercise 1

In this exercise, you'll compute Euler characteristic using ideas from Morse theory. Consider the following picture of Σ_2, a closed, orientable surface of genus 2:



(a) Draw picture of Σ_2 similar to the above, and then imagine it as a two-holed donut with glaze dripping down from the top. Identify the *critical points* on the surface, i.e. points where some particles of the glaze do not flow. Label these critical points with variable names $c_1, c_2, \ldots$.

(b) For each c_j, identify the number of *stable dimensions*, i.e. directions where glaze is flowing downward toward c_j, and also the number of *unstable dimensions*, i.e. directions where glaze is flowing downward away from c_j.

Note: "Directions" here is imprecise, because we really just want to choose two linearly independent directions at each c_j. The number if "directions" is obviously infinite!

(c) For a critical point c_j, the *index* of c_j, denoted $\text{index}(c_j)$, is the number of unstable dimensions at c_j. Write down the index of each critical point. Then, compute the Euler characteristic of Σ_2 using the formula

$$\chi(\Sigma_2) = \sum_i (-1)^i \#\{c_j \ : \ \text{index}(c_j) = i\}.$$

Check for yourself that this agrees with our previous definition of χ using genus.

Exercise 2

In class, we defined the mathematical object known as a *group*. Let g be an element in a group $(G, \cdot)$, where $\cdot$ is the group's binary operation. We think of $\cdot$ like multiplication; for example, we write g^3 for $g \cdot g \cdot g$, and g^{-2} for $g^{-1} \cdot g^{-1}$.

Definition. The *order* of g is

$$\mathrm{ord}(g) = \min\{n \in \mathbb{N} \ : \ g^n = e\},$$

where $e \in G$ is the identity element. For example, $\mathrm{ord}(e) = 1$. If $g^n \neq e$ for all n, then $\mathrm{ord}(g) = \infty$.

(a) Let K be an amphicheiral knot that is *not slice*. Prove that, in the knot concordance group, the equivalence class $[K]$ of K is an order 2 element, i.e. $\mathrm{ord}([K]) = 2$.

(b) Draw a sequence of sketches (i.e. a movie) depicting the *concordance* from the unknot to $4_1\#4_1$. (Hint: You get to choose which diagrams of 4_1 to connect sum.) This shows that $[4_1] \cdot [4_1] = [U]$ in the knot concordance group.

Exercise 3

This exercise will be graded for effort. The goal is to prime you for a more topological construction that we will discuss soon in class.

In this exercise, you'll think about 2×2 matrices as morphisms of a simple category, and then build a bigger category out of it.

First, let's define a category $\mathcal{C}$ where

- there is one object called $\mathbb{R}^2 \in \mathrm{Ob}(\mathcal{C})$, and
- the only morphism set is $\mathrm{Mor}_{\mathcal{C}}(\mathbb{R}^2, \mathbb{R}^2)$, the set of linear transformations from $\mathbb{R}^2$ to itself.

In order to get a handle on this category, let's choose the standard basis for $\mathbb{R}^2$, $\{e_1, e_2\}$. Then, in terms of this basis, we can write

$$\mathrm{Mor}_{\mathcal{C}}(\mathbb{R}^2, \mathbb{R}^2) = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \ : \ a, b, c, d \in \mathbb{R} \right\}.$$

We now define a new category $\mathrm{Mat}(\mathcal{C})$, where the objects are *direct sums* of copies of objects of $\mathcal{C}$, which we now explain. Recall that we decided $\mathbb{R}^2$ is given as the span of the vectors e_1 and e_2: we write

$$\mathbb{R}^2 = \langle e_1, e_2 \rangle,$$

knowing that $e_1 \perp e_2$. Imagine we now have two copies of $\mathbb{R}^2$,

$$\mathbb{R}^2_{(1)} = \langle e_1^{(1)}, e_2^{(1)} \rangle \qquad \text{and} \qquad \mathbb{R}^2_{(2)} = \langle e_1^{(2)}, e_2^{(2)} \rangle.$$

Then their direct sum is

$$\mathbb{R}^2_{(1)} \oplus \mathbb{R}^2_{(2)} = \langle e_1^{(1)}, e_2^{(1)}, e_1^{(2)}, e_2^{(2)} \rangle,$$

with the basis in that particular order, and where we imagine all four basis vectors as mutually orthogonal.

Then $\mathbb{R}^2_{(1)} \oplus \mathbb{R}^2_{(2)} \in \mathrm{Ob}(\mathrm{Mat}(\mathcal{C}))$. You can then direct sum more copies of $\mathbb{R}^2$'s to get vector spaces of any even dimension. But in this category, we really imagine all the copies of $\mathbb{R}^2$ (and direct sums of them) as different objects. For example, $\mathbb{R}^2_{(1)}$ and $\mathbb{R}^2_{(2)}$ are themselves objects in $\mathrm{Mat}(\mathcal{C})$, and they are not the same object.

Now let's consider morphisms. For any 2×2 matrices A and B (i.e. $A, B \in \mathrm{Mor}_{\mathcal{C}}(\mathbb{R}^2, \mathbb{R}^2)$), we can view A as a morphism $\mathbb{R}^2_{(1)} \to \mathbb{R}^2_{(1)}$, and B as a morphism $\mathbb{R}^2_{(1)} \to \mathbb{R}^2_{(2)}$. Then the block matrix

$$\begin{bmatrix} A & B \end{bmatrix}$$

is a morphism $\mathbb{R}^2_{(1)} \oplus \mathbb{R}^2_{(2)} \to \mathbb{R}^2_{(1)}$, i.e.

$$\begin{bmatrix} A & B \end{bmatrix} \in \mathrm{Mor}_{\mathrm{Mat}(\mathcal{C})}\left(\mathbb{R}^2_{(1)} \oplus \mathbb{R}^2_{(2)}, \mathbb{R}^2_{(1)}\right).$$

(a) Read the whole background to this problem; we will later need this example to talk about a more topological category constructed in this way. Then, write down one question you have about this.

(b) Let $V = \mathbb{R}^2_{(1)} \oplus \mathbb{R}^2_{(2)}$. Describe the set $\mathrm{Mor}_{\mathrm{Mat}(\mathcal{C})}(V, V)$. What is the identity morphism?

(c) Suppose U is 2-dimensional, V is 6-dimensional, and W is 4-dimensional. Let A be a morphism $U \to V$, and let B be a morphism $V \to W$. Describe what A and B look like, using $A_1, A_2, \ldots$ and $B_1, B_2, \ldots$ to denote 2×2 matrices in $\mathrm{Mor}_{\mathcal{C}}(\mathbb{R}^2, \mathbb{R}^2)$.

(d) Write down the composition $B \circ A \in \mathrm{Mor}_{\mathrm{Mat}(\mathcal{C})}(U, W)$ is, in terms the matrices A_i, B_i you used in the previous part.